Time step criterion for surface tension applications

1 Simple heat diffusion equation Equation Section 1

Let us suppose an absolutely flat surface of FPM particles. At one arbitrary particle, let us furthermore assume a disturbance of D, i.e. the particle is by D high/lower than all the other particles.

The curvature the is produced by the disturbance D is given by

$$\kappa = \frac{D \cdot \nu}{H^2} \tag{1.1}$$

We estimate the pressure gradient that arises out of this by

$$\frac{1}{\rho}\frac{\partial p}{\partial n} = \frac{1}{\rho}\frac{\sigma\kappa}{B \cdot H} \tag{1.2}$$

Let us suppose the particle cloud is in complete silence, then the implicit scheme allows us to explicitly write down the velocity at the end of the time step with size Δt :

$$\frac{V}{\Delta t} = -\frac{\sigma \cdot \kappa}{\rho \cdot B \cdot H} - \frac{\eta}{\rho} \cdot V \cdot \frac{C}{H^2} \tag{1.3}$$

Using (1.1) we obtain

$$\frac{V}{\Delta t} = -\frac{\sigma \cdot D \cdot \nu}{\rho \cdot B \cdot H^3} - \frac{\eta}{\rho} \cdot V \cdot \frac{C}{H^2} \tag{1.4}$$

Simplification:

$$
\begin{aligned}
V &= -\Delta t \frac{\sigma \cdot D \cdot v}{\rho \cdot B \cdot H^3} - \Delta t \frac{\eta}{\rho} \cdot \frac{C}{H^2} \cdot V \\
V\left(1 + \Delta t \frac{\eta}{\rho} \cdot \frac{C}{H^2}\right) &= -\Delta t \frac{\sigma \cdot D \cdot v}{B \cdot \rho \cdot H^3} \\
V \frac{1}{\rho H^2}\left(\rho H^2 + \Delta t \cdot \eta C\right) &= -\Delta t \frac{\sigma \cdot D \cdot v}{B \cdot \rho \cdot H^3} \\
V \cdot \left(\rho H^2 + \Delta t \cdot \eta C\right) &= -\Delta t \frac{\sigma \cdot D \cdot v}{B \cdot H} \\
V &= -\Delta t \frac{\sigma \cdot D \cdot v}{B \cdot \left(\rho H^3 + \Delta t \cdot \eta H C\right)}
\end{aligned}
\tag{1.5}
$$

We require that, during the time step, the particle will finally not have to move more than a distance of $A \cdot D$, i.e.

$$
|V| < \frac{A \cdot |D|}{\Delta t} \tag{1.6}
$$

Together with (1.5), this means

$$
\Delta t \frac{\sigma \cdot |D| \cdot v}{B \cdot \left(\rho H^3 + \Delta t \cdot \eta H C\right)} < \frac{A \cdot |D|}{\Delta t} \tag{1.7}
$$

Modifications:

$$
\begin{aligned}
&\Delta t \frac{\sigma \cdot v}{B \cdot \left(\rho H^3 + \Delta t \cdot \eta H C\right)} < \frac{A}{\Delta t} \\
&\Delta t^2 \sigma \cdot v < AB \cdot \left(\rho H^3 + \Delta t \cdot \eta H C\right) \\
&\Delta t^2 \sigma \cdot v - \Delta t \cdot ABC \cdot \eta H - AB \cdot \rho H^3 < 0 \\
&\Delta t^2 - \Delta t \cdot \frac{ABC}{\sigma \cdot v} \cdot \eta H - \frac{AB}{\sigma \cdot v} \cdot \rho H^3 < 0 \\
&\left(\Delta t - \frac{1}{2} \cdot \frac{ABC}{\sigma \cdot v} \cdot \eta H\right)^2 - \left(\frac{1}{2} \cdot \frac{ABC}{\sigma \cdot v} \cdot \eta H\right)^2 - \frac{AB}{\sigma \cdot v} \cdot \rho H^3 < 0 \\
&\left(\Delta t - \frac{1}{2} \cdot \frac{ABC}{\sigma \cdot v} \cdot \eta H\right) < \sqrt{\left(\frac{1}{2} \cdot \frac{ABC}{\sigma \cdot v} \cdot \eta H\right)^2 + \frac{AB}{\sigma \cdot v} \cdot \rho H^3}
\end{aligned}
\tag{1.8}
$$

Which finally leads to the desired time step size limiter

$$
\Delta t < \frac{1}{2} \cdot \frac{ABC}{\sigma \cdot v} \cdot \eta H + \sqrt{\left(\frac{1}{2} \cdot \frac{ABC}{\sigma \cdot v} \cdot \eta H\right)^2 + \frac{AB}{\sigma \cdot v} \cdot \rho H^3} \tag{1.9}
$$

A: partition of D, desired distance of the particle to travel during the time step

B: partition of H, distance along which the pressure drops to zero

C: size of central element of Laplace operator: 2 … 10

H: local distance between particles

σ: surface tension

ν : number of space dimensions minus 1

η: viscosity

ρ: density