Coupled linear system of equations (velocity-pressure-coupling)

1 Model for momentum equations

The forward equation for the momentum is

$$\frac{\hat{\mathbf{v}}^{n+1}-\mathbf{v}^{n}}{\Delta t}+\frac{1}{\rho}\nabla p^{n}=\frac{1}{\rho}\nabla \mathbf{S}_{s}+\frac{1}{\rho}\nabla \mathbf{S}_{v}\left(\hat{\mathbf{v}}^{n+1}\right)+\mathbf{g}-\beta\cdot\hat{\mathbf{v}}^{n+1} \tag{1.1}$$

where the pressure of the previous time step p^n is used. This leads to a velocity field, which might not have the correct value for the velocity-divergence $\nabla^T\hat{\mathbf{v}}^{n+1}$. Thus, a correction term c, not known a-priori, is needed. The equation with respect to the correction term is

$$\frac{\mathbf{v}^{n+1}-\mathbf{v}^{n}}{\Delta t}+\frac{1}{\rho}\nabla p^{n}+\frac{1}{\rho}\nabla c=\frac{1}{\rho}\nabla \mathbf{S}_{s}+\frac{1}{\rho}\nabla \mathbf{S}_{v}\left(\mathbf{v}^{n+1}\right)+\mathbf{g}-\beta\cdot\mathbf{v}^{n+1} \tag{1.2}$$

The additional correction term $\frac{1}{\rho}\nabla c$ is supposed to reduce/remove the inconsistent values of $\nabla^T\hat{\mathbf{v}}^{n+1}$.

2 Model for correction pressure

Subtracting equation (1.1) from equation (1.2) yields

$$\frac{\mathbf{v}^{n+1}-\hat{\mathbf{v}}^{n+1}}{\Delta t}+\frac{1}{\rho}\nabla c=\frac{1}{\rho}\left(\nabla \mathbf{S}_{v}\left(\mathbf{v}^{n+1}\right)-\nabla \mathbf{S}_{v}\left(\hat{\mathbf{v}}^{n+1}\right)\right)-\beta\cdot\left(\mathbf{v}^{n+1}-\hat{\mathbf{v}}^{n+1}\right) \tag{2.1}$$

As we might have high non-linearities in the viscous part, i.e. $\nabla \mathbf{S}_{v}\left(\mathbf{v}^{n+1}\right)-\nabla \mathbf{S}_{v}\left(\hat{\mathbf{v}}^{n+1}\right)\neq\nabla \mathbf{S}_{v}\left(\mathbf{v}^{n+1}-\hat{\mathbf{v}}^{n+1}\right)$, we assume that, instead of using the term $\nabla \mathbf{S}_{v}\left(\mathbf{v}^{n+1}\right)-\nabla \mathbf{S}_{v}\left(\hat{\mathbf{v}}^{n+1}\right)$, we can work with the penalty formulation using the virtual time step size (proof not given here), i.e.

$$\left(\mathbf{v}^{n+1}-\hat{\mathbf{v}}^{n+1}\right)+\frac{\Delta t_{v}}{\rho}\nabla c=-\Delta t_{\beta}\beta\cdot\left(\mathbf{v}^{n+1}-\hat{\mathbf{v}}^{n+1}\right) \tag{2.2}$$

Case of Darcy, i.e. $\beta>0$:

We should have $\Delta t_\beta = \Delta t$ in the case of v--, however in the coupled mode vp-, we can prescribe a constant $c_{Darcy}^{\Delta t} = \text{COEFF\_dt\_Darcy}>1$ such that

$$\Delta t_\beta = c_{Darcy}^{\Delta t} \cdot \Delta t \tag{2.3}$$

penalizes wrong values of the divergence of velocity $\nabla^T \mathbf{v}^{n+1}$, see equation (2.10).

Case of viscous flow:

In the same way, Δt_v is to be chosen as small as possible, in fact we have

$$\Delta t_v = \min\left(c_{Viscous}^{\Delta t} \cdot \frac{\rho h^2}{\eta} \ , \ \Delta t \right) \tag{2.4}$$

The term $c_{Viscous}^{\Delta t}$ is adjusted in the input file by the term COEFF_dt_virt.

Equation (2.2) leads to

$$\left(1 + \Delta t_\beta \cdot \beta\right) \cdot \left(\mathbf{v}^{n+1} - \hat{\mathbf{v}}^{n+1}\right) + \frac{\Delta t_v}{\rho} \nabla \varepsilon = 0 \tag{2.5}$$

And hence

$$\left(\mathbf{v}^{n+1} - \hat{\mathbf{v}}^{n+1}\right) + \frac{\Delta t_v}{\left(1 + \Delta t_\beta \cdot \beta\right)} \frac{1}{\rho} \nabla c = 0 \tag{2.6}$$

The virtual time step size is made of the viscosity as well as the Darcy (porous) behaviour of the flow. Therefore we abbreviate

$$\tilde{\Delta} t_v \equiv \frac{\Delta t_v}{\left(1 + \Delta t_\beta \cdot \beta\right)} \tag{2.7}$$

So we have

$$\left(\mathbf{v}^{n+1} - \hat{\mathbf{v}}^{n+1}\right) + \frac{\tilde{\Delta} t_v}{\rho} \nabla c = 0 \tag{2.8}$$

As usual, we apply the divergence operator to this equation in order to obtain a Poisson equation that can be solved by FPM:

$$\nabla^T \left(\mathbf{v}^{n+1} - \hat{\mathbf{v}}^{n+1}\right) + \nabla^T \left(\frac{\tilde{\Delta} t_v}{\rho} \nabla c \right) = 0 \tag{2.9}$$

Finally of course, we have to make use of the fact that we want to set forth a certain divergence of velocity, i.e.

$$\nabla^T \mathbf{v}^{n+1} - \nabla^T \hat{\mathbf{v}}^{n+1} + \nabla^T \left(\frac{\tilde{\Delta} t_v}{\rho} \nabla c \right) = 0 \tag{2.10}$$

For incompressible flows, we require $\nabla^T \mathbf{v}^{n+1} = 0$, in general however we can prescribe any expansion or compression rate of the flow, such that $\nabla^T \mathbf{v}^{n+1}$ is not necessarily zero, such that we can write

$$-\nabla^T \hat{\mathbf{v}}^{n+1} + \nabla^T \left(\frac{\tilde{\Delta} t_v}{\rho} \nabla c \right) = -\nabla^T \mathbf{v}^{n+1}\Big|_{desired} \tag{2.11}$$

3 Coupled system

3.1 Interior FPM-points

Now, equations (1.1) and (2.11) are used to establish the coupled system of linear equations, i.e.

$$\begin{cases} \dfrac{\mathbf{v}^{n+1} - \mathbf{v}^n}{\Delta t} + \dfrac{1}{\rho}\nabla p^n + \dfrac{1}{\rho}\nabla c = \dfrac{1}{\rho}\nabla \mathbf{S}_s + \dfrac{1}{\rho}\nabla \mathbf{S}_v\left(\mathbf{v}^{n+1}\right) + \mathbf{g} - \beta \cdot \mathbf{v}^{n+1} \\ \nabla^T \mathbf{v}^{n+1}\Big|_{desired} - \nabla^T \mathbf{v}^{n+1} + \nabla^T \left(\dfrac{\tilde{\Delta} t_v}{\rho} \nabla c \right) = 0 \end{cases} \tag{3.1}$$

In order to explicitly write the linear equations, we first bring all unknowns left and all known terms to the right of the equal sign

$$\begin{cases} \mathbf{v}^{n+1} + \Delta t \beta \cdot \mathbf{v}^{n+1} - \dfrac{\Delta t}{\rho}\nabla \mathbf{S}_v\left(\mathbf{v}^{n+1}\right) + \dfrac{\Delta t}{\rho}\nabla c = \mathbf{v}^n + \Delta t \cdot \mathbf{g} - \dfrac{\Delta t}{\rho}\nabla p^n + \dfrac{\Delta t}{\rho}\nabla \mathbf{S}_s \\ -\nabla^T \mathbf{v}^{n+1} + \nabla^T \left(\dfrac{\tilde{\Delta} t_v}{\rho} \nabla c \right) = -\nabla^T \mathbf{v}^{n+1}\Big|_{desired} \end{cases} \tag{3.2}$$

We resolve the viscous forces by the linearization

$$\begin{cases} \mathbf{v}^{n+1} + \Delta t \beta \cdot \mathbf{v}^{n+1} - \dfrac{\Delta t}{\rho}\nabla^T\left(\eta \nabla \mathbf{v}^{n+1}\right) + \dfrac{\Delta t}{\rho}\nabla c = \mathbf{v}^n + \Delta t \cdot \mathbf{g} - \dfrac{\Delta t}{\rho}\nabla p^n + \dfrac{\Delta t}{\rho}\nabla \mathbf{S}_s \\ -\nabla^T \mathbf{v}^{n+1} + \nabla^T \left(\dfrac{\tilde{\Delta} t_v}{\rho} \nabla c \right) = -\nabla^T \mathbf{v}^{n+1}\Big|_{desired} \end{cases} \tag{3.3}$$

Here, the local (linearized) viscosity is η.

Now the numerical, meshfree, differential operators are introduced:

$$\begin{cases} \mathbf{v}_i^{n+1} + \Delta t \beta \cdot \mathbf{v}_i^{n+1} - \frac{\Delta t}{\rho} \sum_{j=1}^{M(i)} d_{iJ}^{\nabla^T(\eta\nabla)} \mathbf{v}_J^{n+1} + \frac{\Delta t}{\rho} \sum_{j=1}^{M(i)} d_{iJ}^{\nabla} c_J = \mathbf{v}^n + \Delta t \cdot \mathbf{g} - \frac{\Delta t}{\rho} \tilde{\nabla} p^n + \frac{\Delta t}{\rho} \tilde{\nabla} \mathbf{S}_s \\ -\sum_{j=1}^{M(i)} \left(d_{ij}^{\nabla^T} \cdot \mathbf{v}_J^{n+1} \right) + \sum_{j=1}^{M(i)} \left(d_{iJ}^{\nabla^T\left(\frac{\tilde{\Delta} t_v}{\rho}\nabla\right)} \cdot c_J^{n+1} \right) = -\nabla^T \mathbf{v}^{n+1} \Big|_{desired} \end{cases} \tag{3.4}$$

$M(i)$ is the number of neighbors (i.e. number of nonzero entries in the differential operator), $J = J(j)$ is the global index of the j-th neighbour particle of particle i. Usually, the particle i itself is the first entry in the neighbour list, such that $J(1) = i$.

Finally, we introduce a scaling for the velocity as well as for the correction pressure, i.e.

$$\begin{aligned} \mathbf{v} &= V^0 \cdot \overline{\mathbf{v}} \\ p &= P^0 \cdot \overline{p} \end{aligned} \tag{3.5}$$

In this way, we have representative values V^0, P^0 for velocity and pressure, and the solution to the linear system delivers the non-dimensionalized terms $\overline{\mathbf{v}}, \overline{p}$. The factors V^0, P^0 can be adjusted such that the linear system is nicely configured.

$$\begin{cases} V_i^0 \overline{\mathbf{v}}_i^{n+1} + \Delta t \beta \cdot V_i^0 \overline{\mathbf{v}}_i^{n+1} - \frac{\Delta t}{\rho} \sum_{j=1}^{M(i)} d_{ij}^{\nabla^T(\eta\nabla)} \cdot V_J^0 \overline{\mathbf{v}}_J^{n+1} + \frac{\Delta t}{\rho} \sum_{j=1}^{M(i)} d_{ij}^{\nabla} \cdot P_J^0 \overline{c}_J = \mathbf{v}^n + \Delta t \cdot \mathbf{g} - \frac{\Delta t}{\rho} \tilde{\nabla} p^n + \frac{\Delta t}{\rho} \tilde{\nabla} \mathbf{S}_s \\ -\sum_{j=1}^{M(i)} \left(d_{ij}^{\nabla^T} \cdot V_J^0 \overline{\mathbf{v}}_J^{n+1} \right) + \sum_{j=1}^{M(i)} \left(d_{ij}^{\nabla^T\left(\frac{\tilde{\Delta} t_v}{\rho}\nabla\right)} \cdot P_J^0 \overline{c}_J \right) = -\nabla^T \mathbf{v}^{n+1} \Big|_{desired} \end{cases} \tag{3.6}$$

In the special case of LINEQN_solver = 'NATV' (i.e. native system and trying to establish Bram's structure)), we have

$$\begin{aligned} V_i^0 &= 1 \\ P_i^0 &= 1 \end{aligned} \tag{3.7}$$

And the system is rewritten as

$$
\begin{cases}
\frac{\rho}{\Delta t}\mathbf{v}_i^{n+1} + \rho\beta\cdot\mathbf{v}_i^{n+1} - \sum_{j=1}^{M(i)} d_{ij}^{\nabla^T(\eta\nabla)}\cdot\mathbf{v}_J^{n+1} + \sum_{j=1}^{M(i)} d_{ij}^{\nabla}\cdot c_J = \frac{\rho}{\Delta t}\mathbf{v}^n + \rho\cdot\mathbf{g} - \tilde{\nabla}p^n + \tilde{\nabla}\mathbf{S}_s \\
-\sum_{j=1}^{M(i)}\left(d_{ij}^{\nabla^T}\cdot\mathbf{v}_J^{n+1}\right) + \sum_{j=1}^{M(i)}\left(d_{ij}^{\nabla^T\left(\frac{\tilde{\Delta}t_v}{\rho}\nabla\right)}\cdot c_J\right) = -\nabla^T\mathbf{v}^{n+1}\Big|_{desired}
\end{cases}
\quad (3.8)
$$

The entries of the boundary conditions inside of the matrix are adapted, such that they have the appropriate orders of magnitude.

3.2 Boundary Conditions for FPM-points

3.2.1 Free surface

For the velocity, we have two conditions on the tangential stresses, i.e.

$\mathbf{a}^T\cdot\mathbf{S}\left(\mathbf{v}^{n+1}\right)\cdot\mathbf{n} = a,$

$\mathbf{b}^T\cdot\mathbf{S}\left(\mathbf{v}^{n+1}\right)\cdot\mathbf{n} = b$

and as a third condition we require

$\nabla^T\mathbf{v}^{n+1}\Big| = \nabla^T\mathbf{v}^{n+1}\Big|_{desired}$.

That gives the following equations

$$
\begin{aligned}
&D_1\left(\mathbf{a}\bar{\mathbf{a}}^T + \mathbf{b}\bar{\mathbf{b}}^T\right)\cdot\mathbf{v}^{n+1} - \left(D_2\mathbf{a}\mathbf{a}^T + D_2\mathbf{b}\mathbf{b}^T + D_4\mathbf{n}\mathbf{n}^T\right)\cdot\frac{\partial\mathbf{v}^{n+1}}{\partial n} - \left(D_2\mathbf{a}\mathbf{n}^T + D_4\mathbf{n}\mathbf{a}^T\right)\cdot\frac{\partial\mathbf{v}^{n+1}}{\partial a} \\
&-\left(D_2\mathbf{b}\mathbf{n}^T + D_4\mathbf{n}\mathbf{b}^T\right)\cdot\frac{\partial\mathbf{v}^{n+1}}{\partial b} + D_3\cdot\left(\mathbf{a}\frac{\partial c}{\partial a} + \mathbf{b}\frac{\partial c}{\partial b}\right) - D_4\cdot D_5\cdot\mathbf{n}\cdot c = RightHandSide
\end{aligned}
\quad (3.9)
$$

Where

$D_1 = \frac{\delta\mathrm{h}\cdot\rho}{\Delta\mathrm{t}\cdot N}\cdot\bar{\beta}$

$D_2 = \frac{\eta}{N} +$

$D_3 = \frac{\delta\mathrm{h}}{N}$

$D_4 = \alpha\cdot D_2$ weight for incorporating the div(v)=0 constraint into the boundary condition

$D_5 = 0$ only different from zero if compressible material is used

In this aspect, N is a regularization term such that $D_2 = 1$ or, if inertia are dominant, then $D_1 = 1$.

$\mathbf{n}$ = inward pointing normal of boundary

$\mathbf{a} = \begin{pmatrix} a_x & a_y & a_z \end{pmatrix}$ = first tangential direction

$\mathbf{b} = \begin{pmatrix} b_x & b_y & b_z \end{pmatrix}$ = second tangential direction

$\mathbf{n} \perp \mathbf{a} \perp \mathbf{b}$

For the pressure, we simply have a

$$c = RightHandSide \tag{3.10}$$

The differential operators are discretized in the following way

$$\begin{aligned}
\frac{\partial \mathbf{v}^{n+1}}{\partial n} &= \sum_{j=1}^{M(i)} \left(d_{ij}^{\partial/\partial n} \cdot V_J^0 \overline{\mathbf{v}}_J^{n+1} \right) \\
\frac{\partial \mathbf{v}^{n+1}}{\partial a} &= a_x \cdot \sum_{j=1}^{M(i)} \left(d_{ij}^{\partial/\partial x} \cdot V_J^0 \overline{\mathbf{v}}_J^{n+1} \right) + a_y \cdot \sum_{j=1}^{M(i)} \left(d_{ij}^{\partial/\partial y} \cdot V_J^0 \overline{\mathbf{v}}_J^{n+1} \right) + a_z \cdot \sum_{j=1}^{M(i)} \left(d_{ij}^{\partial/\partial z} \cdot V_J^0 \overline{\mathbf{v}}_J^{n+1} \right) \\
\frac{\partial \mathbf{v}^{n+1}}{\partial b} &= b_x \cdot \sum_{j=1}^{M(i)} \left(d_{ij}^{\partial/\partial x} \cdot V_J^0 \overline{\mathbf{v}}_J^{n+1} \right) + b_y \cdot \sum_{j=1}^{M(i)} \left(d_{ij}^{\partial/\partial y} \cdot V_J^0 \overline{\mathbf{v}}_J^{n+1} \right) + b_z \cdot \sum_{j=1}^{M(i)} \left(d_{ij}^{\partial/\partial z} \cdot V_J^0 \overline{\mathbf{v}}_J^{n+1} \right)
\end{aligned} \tag{3.11}$$

Please note that there is a special operator for $\frac{\partial}{\partial n}$. This is to ensure the most possible diagonal dominance. In the phase of establishing the FPM differential operators, on has the chance to incorporate certain stability constraints.

3.2.2 Solid wall

Two different types of walls are implemented: slip and noslip. The slip wall represents a friction boundary condition, that naturally goes over to a noslip condition, if the friction coefficient is turned to infinity.

3.2.2.1 Solid slip wall

The slip boundaries obey conditions on the tangential stresses in the tangential directions $\mathbf{a}$ and $\mathbf{b}$, i.e.

$$\begin{aligned}
&\alpha \cdot \mathbf{a}^T \cdot \mathbf{S} \cdot \mathbf{n} = \mathbf{a}^T \cdot \left(\mathbf{v} - \mathbf{v}_p \right) \\
&\alpha \cdot \mathbf{b}^T \cdot \mathbf{S} \cdot \mathbf{n} = \mathbf{b}^T \cdot \left(\mathbf{v} - \mathbf{v}_p \right) \\
&\mathbf{n}^T \cdot \left(\mathbf{v} - \mathbf{v}_p \right) = P
\end{aligned} \tag{3.12}$$

$\frac{1}{\alpha}$ is the friction coefficient. If zero, then pure slip, if infinity, pure noslip is given.

A third condition is the flux condition

$$\mathbf{n}^T \cdot \left(\mathbf{v} - \mathbf{v}_p\right) = P \tag{3.13}$$

Where P is the penetration flux through the wall.

The boundary conditions constitute in the system

$$\begin{aligned}
&\left(D_1 \cdot \mathbf{a}\bar{\mathbf{a}}^T + D_1 \cdot \mathbf{b}\bar{\mathbf{b}}^T + D_4 \cdot \mathbf{n}\mathbf{n}^T\right)\cdot \mathbf{v}^{n+1} + \left(\tilde{D}_1 \cdot \mathbf{a}\mathbf{a}^T + \tilde{D}_1 \cdot \mathbf{b}\mathbf{b}^T\right)\cdot \mathbf{v}^{n+1} + \\
&\quad \left(-D_2 \cdot \mathbf{a}\mathbf{a}^T - D_2 \cdot \mathbf{b}\mathbf{b}^T + D_5 \cdot \mathbf{n}\mathbf{n}^T\right)\cdot \frac{\partial \mathbf{v}^{n+1}}{\partial n} + \\
&\quad \left(-D_2 \cdot \mathbf{a}\mathbf{n}^T + D_5 \cdot \mathbf{n}\mathbf{a}^T\right)\cdot \frac{\partial \mathbf{v}^{n+1}}{\partial a} + \\
&\quad \left(-D_2 \cdot \mathbf{b}\mathbf{n}^T + D_5 \cdot \mathbf{n}\mathbf{b}^T\right)\cdot \frac{\partial \mathbf{v}^{n+1}}{\partial b} + \\
&D_3 \cdot \left(\mathbf{a}\frac{\partial c}{\partial \mathrm{a}} + \mathbf{b}\frac{\partial c}{\partial b}\right) = RightHandSide
\end{aligned} \tag{3.14}$$

Usually, the vectors $\bar{\mathbf{a}}, \bar{\mathbf{b}}$ are different from $\mathbf{a}, \mathbf{b}$ only if non-isotropic Darcy law is used.

$$D_1 = \left(\frac{\delta h \cdot \rho}{\Delta \mathrm{t}}\left(1 + \Delta \mathrm{t} \cdot \beta\right)\right)\frac{1}{N}$$

$$\tilde{D}_1 = \left(\frac{1}{\hat{\alpha}}\right)\frac{1}{N}$$

$$D_2 = \frac{\eta}{N}$$

$$D_3 = \frac{\delta h}{N}$$

$$N = \max\left(\left(\frac{\delta h \cdot \rho}{\Delta t} + \frac{1}{\hat{\alpha}} + \ \delta h \cdot \rho \cdot \beta\right),\ \left(\eta \cdot \frac{1}{h^2}\right)\right) = \text{normalization factor}$$

$D_4 = 1$, weight how to link in the flux condition with respect to the tangential stress condition

$D_5 = D_4 \cdot \beta \cdot \delta h = 0$, experimental: weight to link in the div(v)=0 condition into the BC at walls

For the pressure, we set forth a Neumann condition.

$$\frac{\partial c}{\partial n} - D_7 \cdot D_6 \cdot c - D_7 \cdot \nabla^T \mathbf{v} = RightHandSide \tag{3.15}$$

$$D_6 = \left(\frac{\rho_p}{\rho \cdot \Delta t} \right),$$ Kompressibility rate. Zero for incompressible flows

$$D_7 = \left(\alpha h \cdot \frac{\rho}{\Delta t_{virt}} \right),$$ Penalty term to correct improper values of div(v)

3.2.2.2 Solid noslip wall

Just look at slip walls (section 3.2.2.1) and let the friction coefficient go to 0 ($\hat{\alpha} \to 0$). The, the only factor that remains is $\tilde{D}_1 \to 1$. All other factors tend to zero. The factors for pressure D_7, D_6 remain untouched.

4 Dimension analysis of the coupled system

By testing we found, that the convergence of the iterative solution of the linear system (BiCGstab) strongly depends on the local choice of V^0, P^0

Let us analyse that orders of magnitude of the matrix entries given by the system (3.6).

The order of magnitude of the matrix entries of the first equation connected to the velocity is Iv, the one connected to the pressure is Ip, the orders of magnitude for the matrix entries of the second equation are IIv and IIp respectively. We have

$$\begin{aligned} Iv &= \left(1 + \frac{\Delta t}{\rho} \eta \frac{C_L}{h^2} + \Delta t \beta \right) \cdot V^0 \\ Ip &= \frac{\Delta t}{\rho} \frac{C_G}{h} \cdot P^0 \\ IIv &= \frac{C_G}{h} \cdot V^0 \\ IIp &= \frac{\tilde{\Delta} t_v}{\rho} \frac{C_L}{h^2} \cdot P^0 \end{aligned} \tag{4.1}$$

$\frac{C_L}{h^2}$ is the order of magnitude of the discrete Laplace operator, $\frac{C_G}{h}$ is the order of magnitude of the discrete gradient operator. In most cases, $C_L = 10...20$, $C_G = 0.5...1.0$

4.1 Conditions for Matrix normalization

4.1.1 Condition 1

In order to adjust V^0, P^0, we can set forth some useful requirements. The first requirement could be:

$$\frac{Iv}{Ip} = \frac{IIp}{IIv} \tag{4.2}$$

which means the ratio for the diagonal submatrices for each of the four blocks (3 velocity components and the correction pressure) with the off-diagonal submatrices should be equal. That yields

$$\left(1 + \frac{\Delta t}{\rho}\eta\frac{C_L}{h^2} + \Delta t\beta\right)\cdot\left(\frac{V^0}{P^0}\right)^2 = \tilde{\Delta}t_v \cdot \Delta t\frac{C_L}{(\rho h)^2} \tag{4.3}$$

4.1.2 Condition 2

A next condition is the fixation of the ratio between the diagonal submatrix and the off-diagonal submatrices for the pressure equation, i.e.

$$\frac{IIp}{IIv} \geq \varepsilon_P \tag{4.4}$$

This yields

$$\varepsilon_P \cdot \left(\frac{C_G}{C_L}\right)\cdot(\rho h)\cdot\frac{V^0}{P^0} \leq \tilde{\Delta}t_v \tag{4.5}$$

4.1.3 Condition 3

The third condition is similar to the second one: fixation of the ratio between the diagonal and the off-diagonal submatrices for the velocity equation:

$$\frac{Iv}{Ip} \geq \varepsilon_V \tag{4.6}$$

Which yields

$$\left(1 + \frac{\Delta t}{\rho}\eta\frac{C_L}{h^2} + \Delta t\beta\right)\cdot\frac{V^0}{P^0} \geq \varepsilon_V \cdot \frac{\Delta t}{\rho}\cdot\frac{C_G}{h} \tag{4.7}$$

4.2 Critical Sizes for the virtual time step size

Taking conditions 2 and 3, i.e. equations (4.5) and (4.7), we find a critical size of the virtual time step

$$\tilde{\Delta}t_v^{23} \geq \varepsilon_P \cdot \varepsilon_V \cdot \frac{\Delta t \dfrac{C_G^2}{C_L}}{\left(1 + \dfrac{\Delta t}{\rho} \eta \dfrac{C_L}{h^2} + \Delta t \beta\right)} \tag{4.8}$$

For conditions 1 and 2 (i.e. equations (4.3) and (4.5), we find the critical time step size to be

$$\tilde{\Delta}t_v^{12} \geq \varepsilon_P \cdot \varepsilon_P \cdot \frac{\Delta t \dfrac{C_G^2}{C_L}}{\left(1 + \dfrac{\Delta t}{\rho} \eta \dfrac{C_L}{h^2} + \Delta t \beta\right)} \tag{4.9}$$

For conditions 1 and 3, i.e. equations (4.3) and (4.7), the critical virtual time step size is

$$\tilde{\Delta}t_v^{13} \geq \varepsilon_V \cdot \varepsilon_V \cdot \frac{\Delta t \dfrac{C_G^2}{C_L}}{\left(1 + \dfrac{\Delta t}{\rho} \eta \dfrac{C_L}{h^2} + \Delta t \beta\right)} \tag{4.10}$$

In order to fulfil all the conditions, the virtual time step size theoretically has to obey all the critical conditions.

Observation 1: However, it turns out that especially condition (4.8) is meaningful. The observation is, that the system still converges, if $\varepsilon_P \cdot \varepsilon_V > 0.1$ and $\varepsilon_P \geq \varepsilon_V$. $\varepsilon_P \leq \varepsilon_V$ did not provide convergence in any case.

Observation 2: It is also obvious, that, the further we tend away from conditions (4.3), the worse is the convergence of the coupled linear system.

Observation 3: In order to fix good convergence, set $\varepsilon_P = \varepsilon_V = 0.3$. For some of the Darcy cases (big Darcy constant), also $\varepsilon_P = 10; \cdot \varepsilon_V = 0.01$ seemed to be a good choice.