Numerical integration of turbulence models

1 K-Epsilon turbulence model

1.1 Differential equations of the k-epsilon model

For the purpose of this paper, we will concentrate on the k-epsilon-turbulence formulation. The model equations are

$$\begin{aligned} \frac{d(\rho k)}{dt} + (\rho k)\nabla^T \mathbf{v} &= \nabla^T\left(\left(\eta + \frac{\eta_{turb}}{\sigma_k}\right)\nabla k\right) - \rho\varepsilon + P_k + P_b \\ \frac{d(\rho\varepsilon)}{dt} + (\rho\varepsilon)\nabla^T \mathbf{v} &= \nabla^T\left(\left(\eta + \frac{\eta_{turb}}{\sigma_\varepsilon}\right)\nabla \varepsilon\right) - C_{2\varepsilon}\rho\frac{\varepsilon^2}{k} + C_{1\varepsilon}\frac{\varepsilon}{k}\cdot\left(P_k + P_b\right) \end{aligned} \tag{1.1}$$

Here, P_k means the turbulent production rate, and it is determined by

$$P_k = \eta_{turb} \cdot \|\mathbf{D}\|_M^2 \tag{1.2}$$

The term $\|\mathbf{D}\|_M$ is the Mises-norm of the matrix of the velocity gradient.

A similar expression, P_b , is dedicated to turbulent buoyancy effects.

The turbulent viscosity is a function of the turbulent quantities k and epsilon, its quantification is

$$\eta_{turb} = \rho \cdot C_\mu \cdot \frac{k^2}{\varepsilon} \tag{1.3}$$

The given constants are $\sigma_k,\ \sigma_\varepsilon,\ C_{2\varepsilon},\ C_{1\varepsilon},\ C_\mu$.

1.2 Numerical evolution scheme and time integration of the k-epsilon model

The numerical evolution scheme is

$$\begin{aligned}
\frac{d(\rho k)}{dt}+(\rho k)\nabla^T\mathbf{v} &= \tilde{\nabla}^T\left(\left(\eta+\frac{\eta_{turb}}{\sigma_k}\right)\tilde{\nabla}k\right) - \rho\varepsilon + P_k + P_b \\
\frac{d(\rho\varepsilon)}{dt}+(\rho\varepsilon)\nabla^T\mathbf{v} &= \tilde{\nabla}^T\left(\left(\eta+\frac{\eta_{turb}}{\sigma_\varepsilon}\right)\tilde{\nabla}\varepsilon\right) - C_{2\varepsilon}\rho\frac{\varepsilon^2}{k} + C_{1\varepsilon}\frac{\varepsilon}{k}\cdot\left(P_k + P_b\right)
\end{aligned} \tag{1.4}$$

which just arises by replacing the spatial derivatives by its FPM-MLS operators.

For better numerical analysis, we can rewrite this scheme by replacing P_k by its formal expression (1.2) together with (1.3) and, for simplicity, omitting the term P_b

$$\begin{aligned}
\frac{d(k)}{dt} &= \frac{1}{\rho}\left(\tilde{\Delta}_{\hat{\eta}}k\right) - \varepsilon + C_\mu\frac{k^2}{\varepsilon}\|\mathbf{D}\|_M^2 \\
\frac{d(\varepsilon)}{dt} &= \frac{1}{\rho}\left(\tilde{\Delta}_{\hat{\eta}}\varepsilon\right) - C_{2\varepsilon}\frac{\varepsilon^2}{k} + C_{1\varepsilon}C_\mu\cdot k\cdot\|\mathbf{D}\|_M^2
\end{aligned} \tag{1.5}$$

From system (1.5), we derive a singularity formulation, which is either

$$\frac{d}{dt}\left(\frac{k}{\varepsilon}\right) = \left(C_{2\varepsilon}-1\right) + C_\mu\left(1-C_{1\varepsilon}\right)\|\mathbf{D}\|_M^2\cdot\left(\frac{k}{\varepsilon}\right)^2 + \frac{1}{\rho}\left(\tilde{\Delta}_{\hat{\eta}}\frac{k}{\varepsilon}\right) \tag{1.6}$$

or

$$\frac{d}{dt}\left(\frac{\varepsilon}{k}\right) = \left(1-C_{2\varepsilon}\right)\cdot\left(\frac{\varepsilon}{k}\right)^2 + C_\mu\left(C_{1\varepsilon}-1\right)\|\mathbf{D}\|_M^2 + \frac{1}{\rho}\left(\tilde{\Delta}_{\hat{\eta}}\left(\frac{\varepsilon}{k}\right)\right) \tag{1.7}$$

If not both values k and ε are zero, we can provide numerical mean values (ref. section 1.3 Analytical evaluation of the mean values of the singular terms)

$$\left.\frac{k}{\varepsilon}\right|_{mean} = \frac{1}{\Delta t}\int_0^{\Delta t}\left(\frac{k}{\varepsilon}\right)dt \tag{1.8}$$

and

$$\left.\frac{\varepsilon}{k}\right|_{mean} = \frac{1}{\Delta t}\int_0^{\Delta t}\left(\frac{\varepsilon}{k}\right)dt \tag{1.9}$$

It remains to provide a possibly precise numerical time integration of the scheme (1.5) where we avoid singularities by using the mean values (1.8) and (1.9). Thus, the numerical evolution scheme is

$$
\begin{aligned}
\frac{d(k)}{dt} &= \frac{1}{\rho}\left(\tilde{\Delta}_{\hat{\eta}} k\right) - \left(\frac{\varepsilon}{k}\bigg|_{mean}\right)\cdot k + C_\eta \cdot \left(\frac{k}{\varepsilon}\bigg|_{mean}\right) k \left\|\mathbf{D}\right\|_M^2 \\
\frac{d(\varepsilon)}{dt} &= \frac{1}{\rho}\left(\tilde{\Delta}_{\hat{\eta}} \varepsilon\right) - C_{2\varepsilon}\cdot\left(\frac{\varepsilon}{k}\bigg|_{mean}\right)\cdot \varepsilon + C_{1\varepsilon} C_\eta \cdot \left(\frac{k}{\varepsilon}\bigg|_{mean}\right)\cdot \varepsilon \cdot \left\|\mathbf{D}\right\|_M^2
\end{aligned}
\tag{1.10}
$$

1.2.1 Implicit/explicit Euler time integration

1.2.1.1Explicit

For the scheme (1.10), we can now apply two schemes that guaranty the positivity of the terms k and ε.

The first scheme is based on semi-implicit first order time stepping, which is

$$
\begin{aligned}
\frac{k^{n+1}-k^n}{\Delta t} &= \frac{1}{\rho}\left(\tilde{\Delta}_{\hat{\eta}} k^n\right) - \left(\frac{\varepsilon}{k}\bigg|_{mean}\right)\cdot k^{n+1} + C_\eta \cdot \left(\frac{k}{\varepsilon}\bigg|_{mean}\right) \left\|\mathbf{D}\right\|_M^2 \cdot k^n \\
\frac{\varepsilon^{n+1}-\varepsilon^n}{\Delta t} &= \frac{1}{\rho}\left(\tilde{\Delta}_{\hat{\eta}} \varepsilon^n\right) - C_{2\varepsilon}\cdot\left(\frac{\varepsilon}{k}\bigg|_{mean}\right)\cdot \varepsilon^{n+1} + C_{1\varepsilon} C_\eta \cdot \left(\frac{k}{\varepsilon}\bigg|_{mean}\right)\cdot \left\|\mathbf{D}\right\|_M^2 \cdot \varepsilon^n
\end{aligned}
\tag{1.11}
$$

Which finally leads to

$$
\begin{aligned}
k^{n+1} + \Delta t\cdot\left(\frac{\varepsilon}{k}\bigg|_{mean}\right)\cdot k^{n+1} &= k^n + \Delta t\cdot C_\eta\cdot\left(\frac{k}{\varepsilon}\bigg|_{mean}\right)\left\|\mathbf{D}\right\|_M^2 \cdot k^n + \Delta t\cdot\frac{1}{\rho}\left(\tilde{\Delta}_{\hat{\eta}} k^n\right) \\
\varepsilon^{n+1} + \Delta t\cdot C_{2\varepsilon}\cdot\left(\frac{\varepsilon}{k}\bigg|_{mean}\right)\cdot \varepsilon^{n+1} &= \varepsilon^n + \Delta t\cdot C_{1\varepsilon} C_\eta\cdot\left(\frac{k}{\varepsilon}\bigg|_{mean}\right)\cdot\left\|\mathbf{D}\right\|_M^2 \cdot \varepsilon^n + \Delta t\cdot\frac{1}{\rho}\left(\tilde{\Delta}_{\hat{\eta}} \varepsilon^n\right)
\end{aligned}
\tag{1.12}
$$

and more easily to

$$
\begin{aligned}
LHS_k \cdot k^{n+1} &= RHS_k \\
LHS_\varepsilon \cdot \varepsilon^{n+1} &= RHS_\varepsilon
\end{aligned}
\tag{1.13}
$$

where

$$
LHS_k \equiv 1 + \Delta t\cdot\left(\frac{\varepsilon}{k}\bigg|_{mean}\right), \qquad RHS_k \equiv \left(1 + \Delta t\cdot C_\eta\cdot\left(\frac{k}{\varepsilon}\bigg|_{mean}\right)\left\|\mathbf{D}\right\|_M^2\right)\cdot k^n + \Delta t\cdot\frac{1}{\rho}\left(\tilde{\Delta}_{\hat{\eta}} k^n\right)
$$

$$
LHS_\varepsilon \equiv 1 + \Delta t\cdot C_{2\varepsilon}\cdot\left(\frac{\varepsilon}{k}\bigg|_{mean}\right), \qquad RHS_\varepsilon \equiv \left(1 + \Delta t\cdot C_{1\varepsilon} C_\eta\cdot\left(\frac{k}{\varepsilon}\bigg|_{mean}\right)\cdot\left\|\mathbf{D}\right\|_M^2\right)\cdot\varepsilon^n + \Delta t\cdot\frac{1}{\rho}\left(\tilde{\Delta}_{\hat{\eta}} \varepsilon^n\right)
$$

1.2.1.2 Implicit

For the implicit scheme, we have to model the viscous term in an implicit way. That is, rewrite equation (1.11) in the following way:

$$
\begin{aligned}
\frac{k^{n+1}-k^{n}}{\Delta t} &= \frac{1}{\rho}\left(\tilde{\Delta}_{\hat{\eta}} k^{n+1}\right) - \left(\left.\frac{\varepsilon}{k}\right|_{mean}\right)\cdot k^{n+1} + C_{\eta}\cdot\left(\left.\frac{k}{\varepsilon}\right|_{mean}\right)\|\mathbf{D}\|_{M}^{2}\cdot k^{n} \\
\frac{\varepsilon^{n+1}-\varepsilon^{n}}{\Delta t} &= \frac{1}{\rho}\left(\tilde{\Delta}_{\hat{\eta}} \varepsilon^{n+1}\right) - C_{2\varepsilon}\cdot\left(\left.\frac{\varepsilon}{k}\right|_{mean}\right)\cdot \varepsilon^{n+1} + C_{1\varepsilon}C_{\eta}\cdot\left(\left.\frac{k}{\varepsilon}\right|_{mean}\right)\cdot\|\mathbf{D}\|_{M}^{2}\cdot \varepsilon^{n}
\end{aligned}
\tag{1.14}
$$

Application of the diffusion operator yields

$$
\begin{aligned}
\frac{k_i^{n+1}-k_i^{n}}{\Delta t} &= \frac{1}{\rho_i^n}\left(\sum c_{ij}^{\Delta_{\hat{\eta}}}\cdot k_j^{n+1}\right) - \left(\left.\frac{\varepsilon}{k}\right|_{mean}\right)\cdot k_i^{n+1} + C_{\eta}\cdot\left(\left.\frac{k}{\varepsilon}\right|_{mean}\right)\|\mathbf{D}\|_{M}^{2}\cdot k_i^{n} \\
\frac{\varepsilon_i^{n+1}-\varepsilon_i^{n}}{\Delta t} &= \frac{1}{\rho_i^n}\left(\sum c_{ij}^{\Delta_{\hat{\eta}}}\cdot \varepsilon_j^{n+1}\right) - C_{2\varepsilon}\cdot\left(\left.\frac{\varepsilon}{k}\right|_{mean}\right)\cdot \varepsilon_i^{n+1} + C_{1\varepsilon}C_{\eta}\cdot\left(\left.\frac{k}{\varepsilon}\right|_{mean}\right)\cdot\|\mathbf{D}\|_{M}^{2}\cdot \varepsilon_i^{n}
\end{aligned}
\tag{1.15}
$$

Bringing the unknowns on the lefthand side yields finally

$$
\begin{aligned}
k_i^{n+1} + \Delta t\left(\left.\frac{\varepsilon}{k}\right|_{mean}\right)\cdot k_i^{n+1} - \frac{\Delta t}{\rho_i^n}\left(\sum c_{ij}^{\Delta_{\hat{\eta}}}\cdot k_j^{n+1}\right) &= k_i^{n} + \Delta t\cdot C_{\eta}\cdot\left(\left.\frac{k}{\varepsilon}\right|_{mean}\right)\|\mathbf{D}\|_{M}^{2}\cdot k_i^{n} \\
\varepsilon_i^{n+1} + \Delta t\cdot C_{2\varepsilon}\cdot\left(\left.\frac{\varepsilon}{k}\right|_{mean}\right)\cdot \varepsilon_i^{n+1} + \frac{\Delta t}{\rho_i^n}\left(\sum c_{ij}^{\Delta_{\hat{\eta}}}\cdot \varepsilon_j^{n+1}\right) &= +\varepsilon_i^{n} + \Delta t\cdot C_{1\varepsilon}C_{\eta}\cdot\left(\left.\frac{k}{\varepsilon}\right|_{mean}\right)\cdot\|\mathbf{D}\|_{M}^{2}\cdot \varepsilon_i^{n}
\end{aligned}
\tag{1.16}
$$

Which forms a linear system of equations for k and epsilon (separately) that has to be solved by either of the available solver (BiCGstab(2) or SAMG).

1.2.2 Quasi-analytical time integration

If we model the diffusion terms explicitly, we come up with a quasi-analytical solution. of equations (1.10) , which can be rewritten in a simple way as

$$
\begin{aligned}
\frac{d(k)}{dt} &= A_k\cdot k + B_k \\
\frac{d(\varepsilon)}{dt} &= A_\varepsilon\cdot \varepsilon + B_\varepsilon
\end{aligned}
\tag{1.17}
$$

where

$$A_k \equiv C_\eta \cdot \left(\left. \frac{k}{\varepsilon} \right|_{mean} \right) \|\mathbf{D}\|_M^2 - \left(\left. \frac{\varepsilon}{k} \right|_{mean} \right), \qquad B_k \equiv \frac{1}{\rho}\left(\tilde{\Delta}_{\hat{\eta}} k^n\right)$$

$$A_\varepsilon \equiv + C_{1\varepsilon} C_\eta \cdot \left(\left. \frac{k}{\varepsilon} \right|_{mean} \right) \cdot \|\mathbf{D}\|_M^2 - C_{2\varepsilon} \cdot \left(\left. \frac{\varepsilon}{k} \right|_{mean} \right), \qquad B_\varepsilon \equiv \frac{1}{\rho}\left(\tilde{\Delta}_{\hat{\eta}} \varepsilon\right)$$

The analytical solution of (1.17) is given by

$$\begin{aligned} k &= \left(k_0 + \frac{B_k}{A_k} \right) \cdot \exp\left(A_k \left(t - t_0 \right) \right) - \frac{B_k}{A_k} \\ \varepsilon &= \left(\varepsilon_0 + \frac{B_\varepsilon}{A_\varepsilon} \right) \cdot \exp\left(A_\varepsilon \left(t - t_0 \right) \right) - \frac{B_\varepsilon}{A_\varepsilon} \end{aligned} \qquad (1.18)$$

This integration scheme is not yet implemented, however the scheme (1.13) is implemented in FLIQUID_KEPSILON_explicit.F

1.2.3 BDF2 algorithm

We start with equation (1.14) of the implicit formulation and bring the time step size to the right hand side:

$$\begin{aligned} k^{n+1} - k^n &= \Delta t \left(\frac{1}{\rho}\left(\tilde{\Delta}_{\hat{\eta}} k^{n+1}\right) - \left(\left. \frac{\varepsilon}{k} \right|_{mean} \right) \cdot k^{n+1} + C_\eta \cdot \left(\left. \frac{k}{\varepsilon} \right|_{mean} \right) \|\mathbf{D}\|_M^2 \cdot k^n \right) \\ \varepsilon^{n+1} - \varepsilon^n &= \Delta t \left(\frac{1}{\rho}\left(\tilde{\Delta}_{\hat{\eta}} \varepsilon^{n+1}\right) - C_{2\varepsilon} \cdot \left(\left. \frac{\varepsilon}{k} \right|_{mean} \right) \cdot \varepsilon^{n+1} + C_{1\varepsilon} C_\eta \cdot \left(\left. \frac{k}{\varepsilon} \right|_{mean} \right) \cdot \|\mathbf{D}\|_M^2 \cdot \varepsilon^n \right) \end{aligned} \qquad (1.19)$$

That resembles to the BDF1-algorithm, as given in for example in
https://en.wikipedia.org/wiki/Backward_differentiation_formula

The step to BDF2 is now simple:

$$\begin{aligned} \hat{C}_2 k^{n+1} + \hat{C}_1 k^n + \hat{C}_0 k^{n-1} &= \Delta t \left(\frac{1}{\rho}\left(\tilde{\Delta}_{\hat{\eta}} k^{n+1}\right) - \left(\left. \frac{\varepsilon}{k} \right|_{mean} \right) \cdot k^{n+1} + C_\eta \cdot \left(\left. \frac{k}{\varepsilon} \right|_{mean} \right) \|\mathbf{D}\|_M^2 \cdot k^n \right) \\ \hat{C}_2 \varepsilon^{n+1} + \hat{C}_1 \varepsilon^n + \hat{C}_0 \varepsilon^{n-1} &= \Delta t \left(\frac{1}{\rho}\left(\tilde{\Delta}_{\hat{\eta}} \varepsilon^{n+1}\right) - C_{2\varepsilon} \cdot \left(\left. \frac{\varepsilon}{k} \right|_{mean} \right) \cdot \varepsilon^{n+1} + C_{1\varepsilon} C_\eta \cdot \left(\left. \frac{k}{\varepsilon} \right|_{mean} \right) \cdot \|\mathbf{D}\|_M^2 \cdot \varepsilon^n \right) \end{aligned} \qquad (1.20)$$

Where $\hat{C}_2 = \frac{1+2w}{1+w}$, $\hat{C}_1 = -\frac{(1+w)^2}{1+w}$, $\hat{C}_0 = \frac{(w^2)}{1+w}$ with the ratio of the time step sizes $w = \frac{t^{n+1}-t^n}{t^n - t^{n-1}}$. If we are on an Euler/ALE-Level, the also transport terms will appear on the right hand side in the sense

$$\hat{C}_2 k^{n+1} + \hat{C}_1 k^n + \hat{C}_0 k^{n-1} = \Delta t \left(\frac{1}{\rho}\left(\tilde{\Delta}_{\hat{\eta}} k^{n+1}\right) - \left(\mathbf{v}^{ALE}\tilde{\nabla}\right)k^{n+1} - \left(\frac{\varepsilon}{k}\bigg|_{mean}\right)\cdot k^{n+1} + C_\eta \cdot \left(\frac{k}{\varepsilon}\bigg|_{mean}\right)\|\mathbf{D}\|_M^2 \cdot k^n \right)$$
$$\hat{C}_2 \varepsilon^{n+1} + \hat{C}_1 \varepsilon^n + \hat{C}_0 \varepsilon^{n-1} = \Delta t \left(\frac{1}{\rho}\left(\tilde{\Delta}_{\hat{\eta}} \varepsilon^{n+1}\right) - \left(\mathbf{v}^{ALE}\tilde{\nabla}\right)\varepsilon^{n+1} - C_{2\varepsilon} \cdot \left(\frac{\varepsilon}{k}\bigg|_{mean}\right)\cdot \varepsilon^{n+1} + C_{1\varepsilon} C_\eta \cdot \left(\frac{k}{\varepsilon}\bigg|_{mean}\right)\cdot\|\mathbf{D}\|_M^2 \cdot \varepsilon^n \right)$$
(1.21)

Out of this, the following linear system arises:

$$\hat{C}_2 k_i^{n+1} - \frac{\Delta t}{\rho}\sum c_{ij}^{\Delta_{\hat{\eta}}} k_j^{n+1} + \Delta t \sum c_{ij}^{\mathbf{v}^{ALE}\tilde{\nabla}} k_j^{n+1} + \Delta t \left(\frac{\varepsilon}{k}\bigg|_{mean}\right)\cdot k_i^{n+1} = -\hat{C}_1 k_i^n - \hat{C}_0 k_i^{n-1} + \Delta t C_\eta \cdot \left(\frac{k}{\varepsilon}\bigg|_{mean}\right)\|\mathbf{D}\|_M^2 \cdot k_i^n$$
$$\hat{C}_2 \varepsilon^{n+1} - \frac{\Delta t}{\rho}\sum c_{ij}^{\Delta_{\hat{\eta}}} \varepsilon_j^{n+1} + \Delta t \sum c_{ij}^{\mathbf{v}^{ALE}\tilde{\nabla}} \varepsilon_j^{n+1} + \Delta t C_{2\varepsilon} \cdot \left(\frac{\varepsilon}{k}\bigg|_{mean}\right)\cdot \varepsilon^{n+1} = -\hat{C}_1 \varepsilon^n - \hat{C}_0 \varepsilon^{n-1} + \Delta t C_{1\varepsilon} C_\eta \cdot \left(\frac{k}{\varepsilon}\bigg|_{mean}\right)\cdot\|\mathbf{D}\|_M^2 \cdot \varepsilon^n$$
(1.22)

1.3 Analytical evaluation of the mean values of the singular terms

The singular terms $\frac{k}{\varepsilon}\bigg|_{mean} = \frac{1}{\Delta t}\int_0^{\Delta t}\left(\frac{k}{\varepsilon}\right)dt$ and $\frac{\varepsilon}{k}\bigg|_{mean} = \frac{1}{\Delta t}\int_0^{\Delta t}\left(\frac{\varepsilon}{k}\right)dt$ need to be evaluated analytically if we assume negligibility of the diffusion term $\frac{1}{\rho}\left(\tilde{\Delta}_{\hat{\eta}}\left(\frac{\varepsilon}{k}\right)\right)$ in the equation (1.6). This equation then reduces to

$$\frac{d}{dt}\left(\frac{k}{\varepsilon}\right) = \left(C_{2\varepsilon} - 1\right) + C_\mu\left(1 - C_{1\varepsilon}\right)\|\mathbf{D}\|_M^2 \cdot \left(\frac{k}{\varepsilon}\right)^2 \tag{1.23}$$

and a bit more simple

$$\frac{d}{dt}x = A - B\cdot x^2 \tag{1.24}$$

where $x = \frac{k}{\varepsilon}$, $A = \left(C_{2\varepsilon} - 1\right)$, and $B = C_\mu\left(C_{1\varepsilon} - 1\right)\|\mathbf{D}\|_M^2$

The analytical solution to the differential equation (1.24) is

$$\left(\frac{1}{\sqrt{AB}}\cdot arc\tanh\left(\sqrt{\frac{B}{A}}x\right)\right)\Bigg|_{x_0}^{x}=t-t_0 \tag{1.25}$$

In the same fashion,

$$\left(\frac{1}{\sqrt{AB}}\cdot arc\coth\left(\sqrt{\frac{B}{A}}x\right)\right)\Bigg|_{x_0}^{x}=t-t_0 \tag{1.26}$$

is also a solution to the same differential equation. Thus the time evolution of x is given by solving (1.25) and (1.26) for the variable x, i.e.

$$x=\sqrt{\frac{A}{B}}\tanh\left(\sqrt{AB}\left(t-t_0\right)+arc\tanh\left(\sqrt{\frac{B}{A}}x_0\right)\right) \tag{1.27}$$

as well as

$$x=\sqrt{\frac{A}{B}}\coth\left(\sqrt{AB}\left(t-t_0\right)+arc\coth\left(\sqrt{\frac{B}{A}}x_0\right)\right) \tag{1.28}$$

The arctanh is defined between -1 and 1, the arccoth is defined from 1 to infinity and -1 to -infinity. Hence, if

$$\sqrt{\frac{B}{A}}x_0>1 \;\Rightarrow\; x_0>\sqrt{\frac{A}{B}} \tag{1.29}$$

then equation (1.28) has to be used, if

$$\sqrt{\frac{B}{A}}x_0<1 \;\Rightarrow\; x_0<\sqrt{\frac{A}{B}} \tag{1.30}$$

then equation (1.27) has to be used, and finally if

$$\sqrt{\frac{B}{A}}x_0=1 \tag{1.31}$$

then the limit of the differential equation is reached and hence

$$x = \sqrt{\frac{A}{B}} \tag{1.32}$$

1.4 Realizable k-epsilon model

For the realizable k-epsilon model, we have an adapted equation for epsilon, as well as a non-constant C_μ

$$\begin{aligned}
\frac{d(\rho k)}{dt} &= \tilde{\nabla}^T \left(\left(\eta_{lam} + \frac{\eta_{turb}}{\sigma_k} \right) \tilde{\nabla} k \right) - \rho\varepsilon + P_k \\
\frac{d(\rho\varepsilon)}{dt} &= \tilde{\nabla}^T \left(\left(\eta_{lam} + \frac{\eta_{turb}}{\sigma_\varepsilon} \right) \tilde{\nabla} \varepsilon \right) - C_{2\varepsilon}\rho \frac{\varepsilon^2}{k + \sqrt{\nu_{lam}\varepsilon}} + C_1 \rho S \varepsilon
\end{aligned} \tag{1.33}$$

With

$$C_1 = \max\left(0.43, \frac{\eta_{k\varepsilon}}{\eta_{k\varepsilon} + 5}\right) \quad \text{with} \quad \eta_{k\varepsilon} = S\frac{k}{\varepsilon} \quad \text{and } S = \sqrt{2S_{ij}S_{ij}} = \|\mathbf{D}\|_M \tag{1.34}$$

Let us rewrite (1.33) such that we can compare this model with the standard k-epsilon model

$$\begin{aligned}
\frac{d(\rho k)}{dt} &= \tilde{\nabla}^T \left(\left(\eta_{lam} + \frac{\eta_{turb}}{\sigma_k} \right) \tilde{\nabla} k \right) - \rho\varepsilon + P_k \\
\frac{d(\rho\varepsilon)}{dt} &= \tilde{\nabla}^T \left(\left(\eta_{lam} + \frac{\eta_{turb}}{\sigma_\varepsilon} \right) \tilde{\nabla} \varepsilon \right) - C_{2\varepsilon}\rho \frac{\varepsilon^2}{k} \frac{1}{1 + \sqrt{\nu_{lam}\frac{\varepsilon}{k^2}}} + C_1 \rho \frac{k}{k} \frac{\nu_{turb} S}{\nu_{turb} S} S\varepsilon
\end{aligned} \tag{1.35}$$

So far, we did not change, only extracted term and multiplied 1 on the RHS of epsilon. Let us reorder the terms

$$\begin{aligned}
\frac{d(\rho k)}{dt} &= \tilde{\nabla}^T \left(\left(\eta + \frac{\eta_{turb}}{\sigma_k} \right) \tilde{\nabla} k \right) - \rho\varepsilon + P_k \\
\frac{d(\rho\varepsilon)}{dt} &= \tilde{\nabla}^T \left(\left(\eta + \frac{\eta_{turb}}{\sigma_\varepsilon} \right) \tilde{\nabla} \varepsilon \right) - C_{2\varepsilon} \frac{1}{1 + \sqrt{C_\mu \frac{\nu_{lam}}{\nu_{turb}}}} \cdot \rho \frac{\varepsilon^2}{k} + C_1 \frac{k}{\nu_{turb} S} \cdot \frac{\varepsilon}{k} \rho \nu_{turb} S^2
\end{aligned} \tag{1.36}$$

In fact we immediately see, that the realizable model comes up with modified constants

$$
\begin{aligned}
C_{1\varepsilon}^{RKE} &= C_1 \frac{k}{\nu_{turb} S} = C_1 \frac{\varepsilon k}{C_\mu k^2 S} = \frac{C_1}{C_\mu} \frac{\varepsilon}{kS} = \frac{C_1}{C_\mu} \frac{1}{\eta_{k\varepsilon}} \\
C_{2\varepsilon}^{RKE} &= C_{2\varepsilon} \frac{1}{1+\sqrt{C_\mu \frac{\nu}{\nu_{turb}}}}
\end{aligned}
\tag{1.37}
$$

Be aware of the computation of C_μ (kind of a nasty algorithm, for which it is difficult to extract any meaning) :

$$
\begin{aligned}
C_\mu &= \frac{1}{A_0 + A_s \frac{kU^*}{\varepsilon}} \\
U^* &= \sqrt{S_{ij} S_{ij} + \tilde{\Omega}_{ij} \tilde{\Omega}_{ij}} \quad \text{currently } \tilde{\Omega}_{ij} \text{ is neglected is verry confusing in literature} \\
A_0 &= 4.04 \cdot \cos(\ ) \\
A_s &= \sqrt{6} \cdot \cos(\phi) \\
\phi &= \frac{1}{3} \cdot \arccos\left(\sqrt{6} W\right) \\
W &= \frac{S_{ij} S_{jk} S_{ki}}{\left(\frac{S}{\sqrt{2}}\right)^3} \\
S &= \|D\|_M
\end{aligned}
\tag{1.38}
$$

Thus, the realizable k-epsilon model has a similar form like the standard model:

$$
\begin{aligned}
\frac{d(\rho k)}{dt} &= \tilde{\nabla}^T \left(\left(\eta + \frac{\eta_{turb}}{\sigma_k} \right) \tilde{\nabla} k \right) - \rho \varepsilon + P_k \\
\frac{d(\rho \varepsilon)}{dt} &= \tilde{\nabla}^T \left(\left(\eta + \frac{\eta_{turb}}{\sigma_\varepsilon} \right) \tilde{\nabla} \varepsilon \right) - C_{2\varepsilon}^{RKE} \rho \frac{\varepsilon^2}{k} + C_{1\varepsilon}^{RKE} \frac{\varepsilon}{k} P_k \\
\eta_{turb} &= \rho C_\mu \frac{k^2}{\varepsilon}
\end{aligned}
\tag{1.39}
$$

And thus, all the analysis and numerics can be done similar to the standard model, obeying the formation rules of $C_{1\varepsilon}^{RKE}$, $C_{2\varepsilon}^{RKE}$

1.5 RNG k-epsilon model

The model definition is

$$\begin{aligned}\frac{d(\rho k)}{dt} &= \tilde{\nabla}^T\left(\left(\eta_{lam}+\frac{\eta_{turb}}{\sigma_k}\right)\tilde{\nabla}k\right) - \rho\varepsilon + P_k \\ \frac{d(\rho\varepsilon)}{dt} &= \tilde{\nabla}^T\left(\left(\eta_{lam}+\frac{\eta_{turb}}{\sigma_\varepsilon}\right)\tilde{\nabla}\varepsilon\right) - C_{2\varepsilon}^{RNG}\rho\frac{\varepsilon^2}{k} + C_{1\varepsilon}^{RNG}\frac{\varepsilon}{k}P_k\end{aligned} \tag{1.40}$$

The model uses a locally adapted value of $C_{2\varepsilon}^{RNG}$ in the sense

$$\begin{aligned}C_{1\varepsilon}^{RNG} &= C_{1\varepsilon} \\ C_{2\varepsilon}^{RNG} &= C_{2\varepsilon} + \frac{C_\mu \eta_{k\varepsilon}^3\left(1-\left(\eta_{k\varepsilon}/\eta_0\right)\right)}{1+\beta\eta_{k\varepsilon}^3} \\ \eta_{k\varepsilon} &= S\frac{k}{\varepsilon}\end{aligned} \tag{1.41}$$

It used adapted (non-standard) constants as well:

$C_\mu = 0.0845$ (standard: 0.09)

$\sigma_k = 0.7194$ (standard: 1.0)

$\sigma_\varepsilon = 0.7194$ (standard: 1.2)

$C_{1\varepsilon} = 1.42$ (standard: 1.44)

$C_{2\varepsilon} = 1.68$ (standard: 1.92)

$\eta_0 = 4.38$

$\beta = 0.012$

1.6 Boundary conditions for solid walls

The solid walls particles in FPM can be treated like interior particles, with one exception: the wall particles are assumed to be shifted to the interior of the flow domain by a small value $\alpha \cdot h$. The value of α is called the wall layer thickness (stored in ind_WallLayer or in the common-variable WallLayer). Thus, in the model (1.10) , the term $\frac{1}{\rho}\left(\tilde{\Delta}_{\hat{\eta}}k\right)$ needs to have an additional contribution, namely the contribution that comes from the fact that the velocity drops to zero exactly at the wall. So, we have the enhanced model

$$\begin{aligned}dA\cdot\beta h\cdot\frac{d(k)}{dt} &= dA\cdot\beta h\cdot\frac{1}{\rho}\left(\tilde{\Delta}_{\hat{\eta}}k\right)\Big|_{tangential} - dA\cdot\beta h\cdot\left(\frac{\varepsilon}{k}\Big|_{mean}\right)\cdot k + dA\cdot\beta h\cdot C_\mu\cdot\left(\frac{k}{\varepsilon}\Big|_{mean}\right)k\|\mathbf{D}\|_M^2 + dA\cdot\frac{\hat{\eta}}{\rho}\cdot\left(\frac{\partial k}{\partial n}-\frac{k}{\alpha\cdot h}\right) \\ dA\cdot\beta h\cdot\frac{d(\varepsilon)}{dt} &= dA\cdot\beta h\cdot\frac{1}{\rho}\left(\tilde{\Delta}_{\hat{\eta}}\varepsilon\right)\Big|_{tangential} - dA\cdot\beta h\cdot C_{2\varepsilon}\cdot\left(\frac{\varepsilon}{k}\Big|_{mean}\right)\cdot\varepsilon + dA\cdot\beta h\cdot C_{1\varepsilon}C_\mu\cdot\left(\frac{k}{\varepsilon}\Big|_{mean}\right)\cdot\varepsilon\cdot\|\mathbf{D}\|_M^2 + dA\cdot\frac{\hat{\eta}}{\rho}\cdot\left(\frac{\partial\varepsilon}{\partial n}-\frac{\varepsilon}{\alpha\cdot h}\right)\end{aligned} \tag{1.42}$$

dA is the representative area

βh is the control thickness at the boundary (**aware: it is not the layer thickness** αh)

$$\frac{d(k)}{dt} = \frac{1}{\rho}\left(\tilde{\Delta}_{\hat{\eta}}k\right)\Big|_{\text{tangential}} - \left(\frac{\varepsilon}{k}\Big|_{mean}\right)\cdot k + C_\mu \cdot \left(\frac{k}{\varepsilon}\Big|_{mean}\right) k\left\|\mathbf{D}\right\|_M^2 + \frac{\hat{\eta}}{\rho}\cdot\left(\frac{\partial k}{\partial n} - \frac{k}{\alpha \cdot h}\right)\frac{1}{\beta h}$$

$$\frac{d(\varepsilon)}{dt} = \frac{1}{\rho}\left(\tilde{\Delta}_{\hat{\eta}}\varepsilon\right)\Big|_{\text{tangential}} - C_{2\varepsilon}\cdot\left(\frac{\varepsilon}{k}\Big|_{mean}\right)\cdot\varepsilon + C_{1\varepsilon}C_\mu\cdot\left(\frac{k}{\varepsilon}\Big|_{mean}\right)\cdot\varepsilon\cdot\left\|\mathbf{D}\right\|_M^2 + \frac{\hat{\eta}}{\rho}\cdot\left(\frac{\partial\varepsilon}{\partial n} - \frac{\varepsilon}{\alpha\cdot h}\right)\frac{1}{\beta h}$$

In reference to the FPM-code, we call

$$\begin{aligned} \frac{d(k)}{dt}\Big|_{\text{add}} &= \frac{\hat{\eta}}{\rho}\left(\frac{\partial k}{\partial n} - \frac{k}{\alpha\cdot h}\right)\cdot\frac{1}{\beta h} \\ \frac{d(\varepsilon)}{dt}\Big|_{\text{add}} &= \frac{\hat{\eta}}{\rho}\left(\frac{\partial\varepsilon}{\partial n} - \frac{\varepsilon}{\alpha\cdot h}\right)\cdot\frac{1}{\beta h} \end{aligned} \tag{1.43}$$

Moreover, also the production rate P_k has to be extended by a term, which is in the order of magnitude of

$$P_k\Big|_{\text{add}} = \hat{\eta}\cdot\left(\frac{\left\|\left(\mathbf{v}-\mathbf{v}_p\right)-\left(\left(\mathbf{v}-\mathbf{v}_p\right)^T\cdot\mathbf{n}\right)\cdot\mathbf{n}\right\|}{\alpha h}\right)^2 \tag{1.44}$$

where $\mathbf{v}$ = velocity, $\mathbf{v}_p$ =velocity of the wall movement, $\mathbf{n}$= boundary normal

1.7 Incorporation of turbulent wall tension into the velocity boundary conditions

The turbulent wall stress is computed by the following set of equations:

$$U^* = \frac{1}{\kappa}\cdot\ln\left(E\cdot y^*\right) \tag{1.45}$$

$$y^* \equiv \frac{\rho\cdot c_\mu^{1/4}\cdot k_p^{1/2}\cdot y_p}{\eta} \tag{1.46}$$

where the definitions of the wall stress is given by

$$\boldsymbol{\tau}_W \equiv \frac{\rho\cdot c_\mu^{1/4}\cdot k_p^{1/2}}{U^*}\mathbf{U}_p \tag{1.47}$$

- $\mathbf{U}_p = \mathbf{v}_p - \mathbf{v}_{boundary} - C^{virt} \frac{1}{\rho} \nabla p \cdot \tau_p^{virt}$ is the representative velocity difference between the physical wall speed and the fluid velocity in the distance of y_p to the wall. C^{virt} is simply a switch between equilibrium formulation ($C^{virt} = 0$) and non-equilibrium formulation ($C^{virt} = 1$) of the logarithmic wall layer ansatz.
- $k_p^{1/2}$ is the measured/given value of the turbulent kinetic energy at some location with the distance y_p to the wall. Knowing y^* and U^*, we can, according to (1.47), compute the turbulent wall stress.
- $\tau_p^{virt} = \frac{1}{2}\left(\frac{y_v}{\kappa k_p^{1/2}} \ln\left(\frac{y_p}{y_v}\right) + \frac{y_p - y_v}{\kappa k_p^{1/2}} + \frac{\rho}{\eta} y_v^2 \right)$ is some virtual equilibration time together with the definitions
  $y_v = \frac{\eta}{\rho} \frac{y_v^*}{C_\mu^{1/4} k_p^{1/2}}$ and $y_v^* = 11.225$

We assume the value of y_p to be governed by the non-dimensionalized parameter α (see equations (1.42) ff.). In fact we define

$$y_p \equiv \alpha \cdot h \tag{1.48}$$

with h being the local smoothing length. α is also referred to as "wall layer".
We assume **laminar behaviour** in the case of

$$y^* < 11.225 \tag{1.49}$$

In this case, we simply set

$$U^* = y^* \tag{1.50}$$

and then, the wall stress turns out to be the laminar viscous stress in the sense

$$\boldsymbol{\tau}_W \equiv \eta \frac{1}{y_p} \mathbf{U}_p \tag{1.51}$$

With the known values of η, ρ (Viscosity, density) we are able to resolve for the turbulent boundary tension τ_W. The boundary tension finally will be part of the momentum and energy balance of the sliding wall particles. That is done in the following sense: If $\mathbf{t}=\frac{(\mathbf{v}-\mathbf{v}_p)-\left((\mathbf{v}-\mathbf{v}_p)^T\cdot\mathbf{n}\right)\cdot\mathbf{n}}{\left\|(\mathbf{v}-\mathbf{v}_p)-\left((\mathbf{v}-\mathbf{v}_p)^T\cdot\mathbf{n}\right)\cdot\mathbf{n}\right\|}$ is the direction of the velocity along the wall, then we can write down the momentum balance in this direction by

$$\mathbf{t}^T\cdot\frac{d_p\mathbf{v}}{dt}+(\mathbf{v}-\mathbf{v}_p)^T\nabla(\mathbf{t}^T\cdot\mathbf{v})+\frac{1}{\rho}\mathbf{t}^T\cdot\nabla p=\frac{1}{\rho}\nabla^T\left(\hat{\eta}\cdot\nabla(\mathbf{t}^T\cdot\mathbf{v})\right)+\mathbf{t}^T\cdot\mathbf{g} \qquad (1.52)$$

where

- the viscous term $\nabla^T\left(\hat{\eta}\cdot\nabla(\mathbf{t}^T\cdot\mathbf{v})\right)$ is comprised of the physical and turbulent viscosity,
- $\frac{d_p}{dt}$ is the time change rate on the path of the point
- $(\mathbf{v}-\mathbf{v}_p)^T\nabla$ ist he physical transport due to the fact that the point might not move with physical velocity,
- $\frac{d}{dt}=\frac{d_p}{dt}+(\mathbf{v}-\mathbf{v}_p)^T\nabla$ is, in this context, the substantial (total, Lagrangian) time change rate at some point.

The discretization of equation (1.52), based on a control element thickness (user given) of βh, is straight forward in the sense:

$$\frac{\mathbf{t}^T\cdot\bar{\mathbf{v}}^{n+1}-\mathbf{t}^T\cdot\bar{\mathbf{v}}^n}{\Delta t}+\|\mathbf{v}-\mathbf{v}_p\|\mathbf{t}^T\cdot\nabla(\mathbf{t}^T\cdot\mathbf{v}^{n+1})+\frac{1}{\rho}\mathbf{t}^T\cdot\tilde{\nabla}p=\frac{1}{\rho}\frac{\hat{\eta}\cdot\tilde{\nabla}(\mathbf{t}^T\cdot\mathbf{v}^{n+1})\cdot\mathbf{n}^T-\tau_W}{\beta h}+\mathbf{t}^T\cdot\mathbf{g} \quad (1.53)$$

or equivalently

$$\frac{\mathbf{t}^T\cdot\bar{\mathbf{v}}^{n+1}-\mathbf{t}^T\cdot\bar{\mathbf{v}}^n}{\Delta t}+\|\mathbf{v}^n-\mathbf{v}_p^n\|\frac{\tilde{\partial}}{\tilde{\partial}t}(\mathbf{t}^T\cdot\mathbf{v}^{n+1})+\frac{1}{\rho}\mathbf{t}^T\cdot\tilde{\nabla}p=\frac{1}{\rho}\frac{\hat{\eta}\cdot\frac{\tilde{\partial}}{\tilde{\partial}n}(\mathbf{t}^T\cdot\mathbf{v}^{n+1})-\mathbf{t}^T\boldsymbol{\tau}_W}{\beta h}+\mathbf{t}^T\cdot\mathbf{g} \quad (1.54)$$

$\frac{\tilde{\partial}}{\tilde{\partial}t}$ = numerical transport derivative (usually stabilized by some upwind/reconstruction idea) in the direction of $\mathbf{t}$

$\frac{\tilde{\partial}}{\tilde{\partial}n}$ = numerical Neumann derivative, usually stabilized by useful constraints, i.e. derivative in the direction of the boundary normal $\mathbf{n}$

$\bar{\mathbf{v}}$ is the mean velocity in the control thickness βh

which is, in other words, a momentum balance on a control element of the thickness $y_{BND} \equiv \beta \cdot h$ in the direction of $\mathbf{t}$. The term $\hat{\eta} \cdot \frac{\tilde{\partial}}{\tilde{\partial} n}\left(\mathbf{t}^T \cdot \mathbf{v}\right)$ are the viscous stresses on the free-flow-side and τ_W are the viscous stresses at the wall side of the layer-control element. In order to provide a numerically stable formulation, we employ a trick such that τ_W acts as a viscous force that counteracts the velocity, that is

$$\frac{\mathbf{t}^T \cdot \overline{\mathbf{v}}^{n+1} - \mathbf{t}^T \cdot \overline{\mathbf{v}}^n}{\Delta t} + \left\| \mathbf{v}^n - \mathbf{v}_p^n \right\| \frac{\tilde{\partial}}{\tilde{\partial} t}\left(\mathbf{t}^T \cdot \mathbf{v}^{n+1}\right) + \frac{1}{\rho} \mathbf{t}^T \cdot \tilde{\nabla} p = \frac{1}{\rho} \frac{\hat{\eta} \cdot \frac{\tilde{\partial}}{\tilde{\partial} n}\left(\mathbf{t}^T \cdot \mathbf{v}^{n+1}\right)}{\beta h} + \mathbf{t}^T \cdot \mathbf{g} - \frac{1}{\beta h} \frac{1}{\rho} \frac{\tau_W}{\left\| \mathbf{t}^T \cdot \mathbf{v}^* \right\|} \mathbf{t}^T \cdot \mathbf{v}^{n+1}$$

As $\overline{\mathbf{v}}$ is the mean velocity, it is not exactly the velocity $\mathbf{v}$ at the virtual location $y_p \equiv \alpha \cdot h$ (see equation (1.48)) a bit away from the wall. So, we need a correction factor $\overline{\mathbf{v}} = A \cdot \mathbf{v}$ that maps the mean velocity towards the true velocity at . So, the equation to solve is actually

$$\frac{\mathbf{t}^T \cdot A\mathbf{v}^{n+1} - \mathbf{t}^T \cdot A\mathbf{v}^n}{\Delta t} + \left\| \mathbf{v}^n - \mathbf{v}_p^n \right\| \frac{\tilde{\partial}}{\tilde{\partial} t}\left(\mathbf{t}^T \cdot \mathbf{v}^{n+1}\right) + \frac{1}{\rho} \mathbf{t}^T \cdot \tilde{\nabla} p = \frac{1}{\rho} \frac{\hat{\eta} \cdot \frac{\tilde{\partial}}{\tilde{\partial} n}\left(\mathbf{t}^T \cdot \mathbf{v}^{n+1}\right)}{\beta h} + \mathbf{t}^T \cdot \mathbf{g} - \frac{1}{\beta h} \frac{1}{\rho} \frac{\tau_W}{\left\| \mathbf{t}^T \cdot \mathbf{v}^* \right\|} \mathbf{t}^T \cdot \mathbf{v}^{n+1}$$

(1.55)

Or in other words

$$\frac{\mathbf{t}^T \cdot \mathbf{v}^{n+1} - \mathbf{t}^T \cdot \mathbf{v}^n}{\overline{\Delta t}} + \left\| \mathbf{v}^n - \mathbf{v}_p^n \right\| \frac{\tilde{\partial}}{\tilde{\partial} t}\left(\mathbf{t}^T \cdot \mathbf{v}^{n+1}\right) + \frac{1}{\rho} \mathbf{t}^T \cdot \tilde{\nabla} p = \frac{1}{\rho} \frac{\hat{\eta} \cdot \frac{\tilde{\partial}}{\tilde{\partial} n}\left(\mathbf{t}^T \cdot \mathbf{v}^{n+1}\right)}{\beta h} + \mathbf{t}^T \cdot \mathbf{g} - \frac{1}{\beta h} \frac{1}{\rho} \frac{\tau_W}{\left\| \mathbf{t}^T \cdot \mathbf{v}^* \right\|} \mathbf{t}^T \cdot \mathbf{v}^{n+1}$$

(1.56)

And we have $\overline{\Delta t} = \frac{\Delta t}{A}$. The representative velocity $\mathbf{v}^*$ would most preferably have to be $\mathbf{v}^{n+1}$, however then the equation becomes nonlinear and difficult to solve numerically. Actually, we choose $\mathbf{v}^* = \mathbf{v}^n$. Finally, the whole solution of the boundary velocity can be rewritten as

$$\left(\frac{1}{\Delta t} + \frac{1}{\beta h} \frac{1}{\rho} \frac{\tau_W}{\left\| \mathbf{t}^T \cdot \mathbf{v}^* \right\|} \right) \mathbf{t}^T \cdot \mathbf{v}^{n+1} + \left\| \mathbf{v}^n - \mathbf{v}_p^n \right\| \frac{\tilde{\partial}}{\tilde{\partial} t}\left(\mathbf{t}^T \cdot \mathbf{v}^{n+1}\right) - \frac{1}{\rho} \frac{\hat{\eta} \cdot \frac{\tilde{\partial}}{\tilde{\partial} n}\left(\mathbf{t}^T \cdot \mathbf{v}^{n+1}\right)}{\beta h} = \frac{\mathbf{t}^T \cdot \mathbf{v}^n}{\Delta t} - \frac{1}{\rho} \mathbf{t}^T \cdot \tilde{\nabla} p + \mathbf{t}^T \cdot \mathbf{g}$$

Last remaining task is to determine the correction factor A.

$$U^* = \begin{cases} \frac{1}{\kappa}\ln\left(E\cdot y^*\right), & y^* > 11.225 \\ y^*, & else \end{cases} \tag{1.57}$$

We have

$$U^{*\prime} = \frac{dU^*}{dy^*} = \min\left(\frac{1}{\kappa}\frac{1}{y^*}, 1\right) \tag{1.58}$$

The min-operator pays attention to the fact, that the maximum derivative can be 1 (in the laminar sublayer). The ratio of the average velocity and the velocity at point $y_p \equiv \alpha\cdot h$ is

$$\begin{aligned} \frac{\bar{U}_L}{U_p} &= \frac{1}{2}\frac{U_0^{*\prime}}{U^*}\frac{x^{*2}}{y^*} + \frac{1}{2}\left(\frac{U_0^{*\prime}}{U^*}x^* + 1\right)\left(1 - \frac{x^*}{y^*}\right) \\ x^* &= \frac{U^* - U^{*\prime}y^*}{U_0^{*\prime} - U^{*\prime}} \end{aligned} \tag{1.59}$$

Right of the point $y_p \equiv \alpha\cdot h$ we have

$$\begin{aligned} \frac{\bar{U}_R}{U_p} &= 1 + \frac{1}{2}\frac{\partial U}{\partial n}\frac{z}{U_p} \\ z &= \left(\beta - \alpha\right)h \end{aligned} \tag{1.60}$$

Finally, we have

$$A = \frac{\bar{U}}{U_p} = \frac{\bar{U}_L}{U_p}\frac{y_p}{\beta h} + \frac{\bar{U}_R}{U_p}\frac{\beta h - y_p}{\beta h} \tag{1.61}$$

2 Simple 1D formulation and test program

2.1 1D formulation

Let us consider the equations for k, epsilon, and velocity as a set of 1D equations:

$$\frac{\partial(\rho k)}{\partial t} = \frac{\partial}{\partial x}\left(\left(\eta + \frac{\eta_{turb}}{\sigma_k}\right)\frac{\partial}{\partial x}k\right) - \rho\varepsilon + P_k$$

$$\frac{\partial(\rho\varepsilon)}{\partial t} = \frac{\partial}{\partial x}\left(\left(\eta + \frac{\eta_{turb}}{\sigma_\varepsilon}\right)\frac{\partial}{\partial x}\varepsilon\right) - C_{2\varepsilon}\rho\frac{\varepsilon^2}{k} + C_{1\varepsilon}\frac{\varepsilon}{k}\cdot P_k \tag{2.1}$$

$$\frac{\partial(\rho u)}{\partial t} = \frac{\partial}{\partial x}\left(\left(\eta + \eta_{turb}\right)\frac{\partial}{\partial x}u\right) + \rho g_x$$

This can be solved numerically on a very simple basis of equidistant numerical points with the distance h.

The turbulent viscosity is given by

$$\eta_{turb} = \rho c_\mu \frac{k^2}{\varepsilon} \tag{2.2}$$

The points on the left and right boundaries will have to fulfil the boundary conditions, which result from the according PDE (2.1).

In detail, for the velocity we have

$$\frac{\partial(\rho u)}{\partial t} = \frac{1}{\beta h}\left(\left(\eta + \eta_{turb}\right)\left(\frac{\partial u}{\partial n} - \frac{u - u^{BND}}{\alpha h}\right)\right) + \rho g_x \tag{2.3}$$

βh is the thickness of the control element around the boundary and

u^{BND} is the velocity value exactly at the boundary.

The term $\left(\eta + \eta_{turb}\right)\left(-\frac{u - u^{BND}}{\alpha h}\right)$ is an expression for the wall stress. Thus, we rather formulate

$$\frac{\partial(\rho u)}{\partial t} = \frac{1}{\beta h}\left(\left(\eta + \eta_{turb}\right)\frac{\partial u}{\partial n} + \tau_W\right) + \rho g_x \tag{2.4}$$

τ_W is the turbulent wall stress, which reduces to the laminar wall stress is turbulence becomes negligible. The turbulent wall stress is subject to different models. The most simple one is

$$\tau_W \equiv \frac{\rho \cdot c_\mu^{1/4} \cdot k_p^{1/2}}{U^*}\left(-\left(u-u^{BND}\right)\right)$$

$$U^* = \begin{cases} \frac{1}{\kappa} \cdot \ln\left(E \cdot y^*\right) & \text{if } y^* > 11.225 \\ y^* & \text{else} \end{cases} \tag{2.5}$$

$$y^* \equiv \frac{\rho \cdot c_\mu^{1/4} \cdot k_p^{1/2} \cdot y_p}{\eta}$$

$$y_p = \alpha h$$

For k we have in a similar fashion

$$\frac{\partial(\rho k)}{\partial t} = \frac{1}{\beta h}\left(\left(\eta + \frac{\eta_{turb}}{\sigma_k}\right)\left(\frac{\partial k}{\partial n} - \frac{k-k^{BND}}{\alpha h}\right)\right) - \rho\varepsilon + \bar{P}_k \tag{2.6}$$

αh is the virtual small distance by which the boundary point is moved into the boundary layer

$k^{BND} = 0$ is the expected value the turbulent kinetic energy shall have exactly at the wall,

$\bar{P}_k$ is the production rate average in the control layer of thickness βh .

Finally, the equation for epsilon is not surprising anymore:

$$\frac{\partial(\rho\varepsilon)}{\partial t} = \frac{1}{\beta h}\left(\left(\eta + \frac{\eta_{turb}}{\sigma_\varepsilon}\right)\left(\frac{\partial \varepsilon}{\partial n} - \frac{\varepsilon-\varepsilon^{BND}}{\alpha h}\right)\right) - C_{2\varepsilon}\rho\frac{\varepsilon^2}{k} + C_{1\varepsilon}\frac{\varepsilon}{k}\cdot\bar{P}_k \tag{2.7}$$

ε^{BND} again is the value of the dissipation rate expected exactly at the boundary.

The logical approach is $\bar{P}_k = P_k$. However, in the Ansys manual, a derived ansatz is provided, which differs.

2.2 Limitation of the numerical layer thickness

From equation (2.1), we can derive a limitation of the layer thickness assumed in a numerical simulation.

Let us assume, there is only laminar viscosity (turbulent viscosity would make the layer bigger). Wlog, let us assume that gravity does not play a role. In this case, the velocity equation would reduce to

$$\frac{\partial(\rho u)}{\partial t} = \eta \frac{\partial}{\partial x}\left(\frac{\partial}{\partial x} u\right) \tag{2.8}$$

The numerical representation of this equation in a finite difference framework for the first point beyond the wall would be

$$\frac{u_1^{n+1} - u_1^n}{\Delta t} = \frac{\eta}{\rho} \frac{1}{\Delta x^2} \left(u_0 - 2u_1^{n+1} + u_2^{n+1}\right) \tag{2.9}$$

Let us assume the flow is initialized with $u_{freeStream}$, the wall is at zero speed, and we consider the first time step, then we can simplify

$$\begin{aligned}
\frac{u_1^{n+1} - u_{freeStream}}{\Delta t} &= \frac{\eta}{\rho} \frac{1}{\Delta x^2} \left(-2u_1^{n+1} + u_{freeStream}\right) \\
\Rightarrow \quad u_1^{n+1} + \frac{\eta}{\rho}\frac{\Delta t}{\Delta x^2} 2u_1^{n+1} &= \frac{\eta}{\rho}\frac{\Delta t}{\Delta x^2} u_{freeStream} + u_{freeStream} \\
\Rightarrow \quad \left(1 + 2\frac{\eta}{\rho}\frac{\Delta t}{\Delta x^2}\right) u_1^{n+1} &= \left(1 + \frac{\eta}{\rho}\frac{\Delta t}{\Delta x^2}\right) u_{freeStream} \qquad (2.10) \\
\Rightarrow \quad \frac{u_1^{n+1}}{u_{freeStream}} &= \frac{\left(1 + \frac{\eta}{\rho}\frac{\Delta t}{\Delta x^2}\right)}{\left(1 + 2\frac{\eta}{\rho}\frac{\Delta t}{\Delta x^2}\right)}
\end{aligned}$$

Let us now assume, for given time step size, we would like to know the discretization size, such that the velocity after the time step drops to a relative level of α compared to the free flow velocity. That is

$$\begin{aligned}
\alpha &= \frac{\left(1 + \frac{\eta}{\rho}\frac{\Delta t}{\Delta x^2}\right)}{\left(1 + 2\frac{\eta}{\rho}\frac{\Delta t}{\Delta x^2}\right)} \\
\Rightarrow \quad \alpha\left(1 + 2\frac{\eta}{\rho}\frac{\Delta t}{\Delta x^2}\right) &= \left(1 + \frac{\eta}{\rho}\frac{\Delta t}{\Delta x^2}\right) \qquad (2.11) \\
\Rightarrow \quad 2\alpha\frac{\eta}{\rho}\Delta t - \frac{\eta}{\rho}\Delta t &= \Delta x^2 - \alpha\Delta x^2 \\
\Rightarrow \quad (2\alpha - 1)\frac{\eta}{\rho}\Delta t &= (1 - \alpha)\Delta x^2
\end{aligned}$$

And so we have

$$\Delta x = \sqrt{\frac{(2\alpha - 1)}{(1-\alpha)}} \sqrt{\frac{\eta}{\rho}\Delta t} = \beta \sqrt{\frac{\eta}{\rho}\Delta t} \qquad (2.12)$$

$\Delta x = \beta \sqrt{\frac{\eta}{\rho}\Delta t}$ **represents the order of magnitude of the minimal layer thickness,** because**:**

- The layer becomes wider in the following time step,
- Viscosity will increase due to turbulent effects.

Choose β reasonably between 0 and 1. Rather tend to 0.1 or so (my feeling).

This layer thickness limit can be used to choose the distance of the first computation point in front of the wall.

Remarks:

- the same estimation one would get when using the analytical solutions given in literature for 1D-heat-transfer problems.
- A dimension analysis of the PDE would reveal the same behavior.

From the estimation of the layer thickness (2.12), we can derive an estimate for the upper limit of the wall stress itself by

$$\begin{aligned} & L > \beta \sqrt{\frac{\eta}{\rho}\Delta t} \\ & \frac{1}{\beta \sqrt{\frac{\eta}{\rho}\Delta t}} > \frac{1}{L} \\ & \frac{\eta u_{freeStream}}{\beta \sqrt{\frac{\eta}{\rho}\Delta t}} > \frac{\eta u_{freeStream}}{L} = \tau_W \end{aligned} \qquad (2.13)$$

However, here I am not sure if turbulent effects might find a mechanism to even produce bigger wall stresses (i.e. steeper gradients of velocity in the laminar region).

3 Further analytical considerations of the k-epsilon model

The following small analysis of the k epsilon model is based on simply the production and dissipation rates of the model, however we neglect the diffusion terms. This is a simplification that signifies a homogeneous turbulence distribution, for instance in a mixing unit without big gradients in the turbulent quantities.

3.1 Reduced model without diffusion

$$\dot{k} = C_\mu \cdot \frac{k^2}{\varepsilon} \cdot G - \varepsilon$$
$$\dot{\varepsilon} = C_{1\varepsilon} \cdot C_\mu \cdot k \cdot G - C_{2\varepsilon} \frac{\varepsilon^2}{k} \tag{3.1}$$

Here, $G = \|\mathbf{D}\|_M^2$. The consideration of the k-epsilon system without viscous terms seems to be useful as we can always imagine cases where k and epsilon are evenly distributed, for instance a liquid in a turbulent mixing unit.

3.2 Limit of the relation k and epsilon

The time derivative of the term $\frac{k}{\varepsilon}$ is given by the scheme above

$$\dot{k} \cdot \varepsilon = C_\mu \cdot k^2 \cdot G - \varepsilon^2$$
$$k \cdot \dot{\varepsilon} = C_{1\varepsilon} \cdot C_\mu \cdot k^2 \cdot G - C_{2\varepsilon} \cdot \varepsilon^2 \tag{3.2}$$

$$\begin{aligned} \frac{d}{dt}\left(\frac{k}{\varepsilon}\right) &= \frac{\dot{k} \cdot \varepsilon - k \cdot \dot{\varepsilon}}{\varepsilon^2} \\ &= \frac{C_\mu \cdot k^2 \cdot G - \varepsilon^2 - C_{1\varepsilon} \cdot c_\mu \cdot k^2 \cdot G + C_{2\varepsilon} \cdot \varepsilon^2}{\varepsilon^2} \\ &= \frac{C_\mu \cdot \left(1 - C_{1\varepsilon}\right) \cdot k^2 \cdot G + \left(C_{2\varepsilon} - 1\right) \cdot \varepsilon^2}{\varepsilon^2} \\ &= C_\mu \cdot \left(1 - C_{1\varepsilon}\right) \cdot \frac{k^2}{\varepsilon^2} \cdot G + \left(C_{2\varepsilon} - 1\right) \end{aligned} \tag{3.3}$$

From equation (3.3) we learn that the limit of the value $\frac{k}{\varepsilon}$ as time goes to infinity is

$$\begin{aligned} 0 &= C_\mu \cdot \left(1 - C_{1\varepsilon}\right) \cdot \frac{k^2}{\varepsilon^2} \cdot G + \left(C_{2\varepsilon} - 1\right) \\ \left(1 - C_{2\varepsilon}\right) &= C_\mu \cdot \left(1 - C_{1\varepsilon}\right) \cdot \frac{k^2}{\varepsilon^2} \cdot G \\ \frac{\left(1 - C_{2\varepsilon}\right)}{\left(1 - C_{1\varepsilon}\right)} \cdot \frac{1}{C_\mu \cdot G} &= \frac{k^2}{\varepsilon^2} \\ \frac{k}{\varepsilon} &= \sqrt{\frac{\left(1 - C_{2\varepsilon}\right)}{\left(1 - C_{1\varepsilon}\right)}} \cdot \sqrt{\frac{1}{C_\mu \cdot G}} \end{aligned} \tag{3.4}$$

3.3 Relation of time change of diffusion

The turbulent diffusion is given by

$$\nu = C_\mu \cdot \frac{k^2}{\varepsilon} \tag{3.5}$$

The time change rate is

$$\begin{aligned}
\dot{\nu} &= C_\mu \cdot \frac{d}{dt}\left(k^2 \cdot \varepsilon^{-1}\right) \\
&= C_\mu \cdot \left(2k \cdot \dot{k} \cdot \varepsilon^{-1} - k^2 \cdot \varepsilon^{-2} \cdot \dot{\varepsilon}\right) \\
&= C_\mu \cdot \left(2k \cdot \varepsilon^{-1} \cdot \left(C_\mu \cdot \frac{k^2}{\varepsilon} \cdot G \; - \; \varepsilon\right) - k^2 \cdot \varepsilon^{-2} \cdot \left(C_{1\varepsilon} \cdot C_\mu \cdot k \cdot G \; - \; C_{2\varepsilon} \frac{\varepsilon^2}{k}\right)\right) \\
&= C_\mu \cdot \left(\left(2c_\mu \cdot \frac{k^3}{\varepsilon^2} \cdot G \; - \; 2k\right) - \left(C_{1\varepsilon} \cdot C_\mu \cdot \frac{k^3}{\varepsilon^2} \cdot G \; - \; C_{2\varepsilon} \cdot k\right)\right) \\
&= C_\mu \cdot \left(\left(2 - C_{1\varepsilon}\right) C_\mu \cdot \frac{k^3}{\varepsilon^2} \cdot G + \left(C_{2\varepsilon} - 2\right) \cdot k\right)
\end{aligned} \tag{3.6}$$

And finally

$$\begin{aligned}
\frac{\dot{\nu}}{\nu} &= \frac{C_\mu \cdot \left(\left(2 - C_{1\varepsilon}\right) C_\mu \cdot \frac{k^3}{\varepsilon^2} \cdot G + \left(C_{2\varepsilon} - 2\right) \cdot k\right)}{c_\mu \cdot \frac{k^2}{\varepsilon}} \\
\frac{\dot{\nu}}{\nu} &= \left(\left(2 - C_{1\varepsilon}\right) \cdot C_\mu \cdot \frac{k}{\varepsilon} \cdot G + \left(C_{2\varepsilon} - 2\right) \cdot \frac{\varepsilon}{k}\right)
\end{aligned} \tag{3.7}$$

So, of course we can also determine the change rate of the turbulent viscosity at infinite time:

$$\begin{aligned}
\frac{\dot{\nu}}{\nu} &= \left(\left(2 - C_{1\varepsilon}\right) \cdot \sqrt{\frac{\left(1 - C_{2\varepsilon}\right)}{\left(1 - C_{1\varepsilon}\right)}} \cdot \sqrt{C_\mu \cdot G} + \left(C_{2\varepsilon} - 2\right) \cdot \sqrt{\frac{\left(1 - C_{1\varepsilon}\right)}{\left(1 - C_{2\varepsilon}\right)}} \cdot \sqrt{C_\mu \cdot G}\right) \\
\frac{\dot{\nu}}{\nu} &= \left(\left(2 - C_{1\varepsilon}\right) \cdot \sqrt{\frac{\left(1 - C_{2\varepsilon}\right)}{\left(1 - C_{1\varepsilon}\right)}} + \left(C_{2\varepsilon} - 2\right) \cdot \sqrt{\frac{\left(1 - C_{1\varepsilon}\right)}{\left(1 - C_{2\varepsilon}\right)}}\right) \cdot \sqrt{C_\mu \cdot G}
\end{aligned} \tag{3.8}$$

We can now also ask for a dedicated change of the turbulent viscosity during a given time step, i.e.

$$\alpha = \frac{\dot{\nu} \cdot \Delta t}{\nu} = \left(\left(2 - C_{1\varepsilon}\right) \cdot C_{\mu} \cdot \frac{k}{\varepsilon} \cdot G + \left(C_{2\varepsilon} - 2\right) \cdot \frac{\varepsilon}{k} \right) \cdot \Delta t \tag{3.9}$$

For given k, we can ask what should be epsilon, ergo

$$\begin{aligned}
\alpha \cdot k \cdot \varepsilon &= \left(2 - C_{1\varepsilon}\right) \cdot C_{\mu} \cdot k^2 \cdot G \cdot \Delta t + \left(C_{2\varepsilon} - 2\right) \cdot \Delta t \cdot \varepsilon^2 \\
0 &= \left(2 - C_{1\varepsilon}\right) \cdot C_{\mu} \cdot k^2 \cdot G \cdot \Delta t - \alpha \cdot k \cdot \varepsilon + \left(C_{2\varepsilon} - 2\right) \cdot \Delta t \cdot \varepsilon^2 \\
0 &= \frac{\left(2 - C_{1\varepsilon}\right) \cdot C_{\mu} \cdot k^2 \cdot G}{\left(C_{2\varepsilon} - 2\right)} - 2 \frac{\alpha \cdot k}{2 \cdot \left(C_{2\varepsilon} - 2\right) \cdot \Delta t} \cdot \varepsilon + \varepsilon^2 \\
0 &= \frac{\left(2 - C_{1\varepsilon}\right) \cdot C_{\mu} \cdot k^2 \cdot G}{\left(C_{2\varepsilon} - 2\right)} - \left(\frac{\alpha \cdot k}{2 \cdot \left(C_{2\varepsilon} - 2\right) \cdot \Delta t} \right)^2 + \left(\varepsilon - \frac{\alpha \cdot k}{2 \cdot \left(C_{2\varepsilon} - 2\right) \cdot \Delta t} \right)^2 \\
0 &= \frac{4 \cdot \left(2 - C_{1\varepsilon}\right) \cdot \left(C_{2\varepsilon} - 2\right) \cdot C_{\mu} \cdot k^2 \cdot \Delta t^2 \cdot G}{\left(2 \cdot \left(C_{2\varepsilon} - 2\right) \cdot \Delta t\right)^2} - \left(\frac{\alpha \cdot k}{2 \cdot \left(C_{2\varepsilon} - 2\right) \cdot \Delta t} \right)^2 + \left(\varepsilon - \frac{\alpha \cdot k}{2 \cdot \left(C_{2\varepsilon} - 2\right) \cdot \Delta t} \right)^2 \\
&\frac{\alpha^2 \cdot k^2 - 4 \cdot \left(2 - C_{1\varepsilon}\right) \cdot \left(C_{2\varepsilon} - 2\right) \cdot C_{\mu} \cdot k^2 \cdot \Delta t^2 \cdot G}{\left(2 \cdot \left(C_{2\varepsilon} - 2\right) \cdot \Delta t\right)^2} = \left(\varepsilon - \frac{\alpha \cdot k}{2 \cdot \left(C_{2\varepsilon} - 2\right) \cdot \Delta t} \right)^2 \\
\varepsilon &= \frac{\alpha \cdot k}{2 \cdot \left(C_{2\varepsilon} - 2\right) \cdot \Delta t} + \sqrt{\frac{\alpha^2 - 4 \cdot \left(2 - C_{1\varepsilon}\right) \cdot \left(C_{2\varepsilon} - 2\right) \cdot C_{\mu} \cdot \Delta t^2 \cdot G}{\left(2 \cdot \left(C_{2\varepsilon} - 2\right) \cdot \Delta t\right)^2}} \cdot k
\end{aligned} \tag{3.10}$$

3.4 Paradoxon of the characteristic turbulent length scale

We remember, from literature, that the turbulent length scale is defined as

$$L = \frac{k^{3/2}}{\varepsilon} \tag{3.11}$$

The time change rate is now easily derived by

$$\dot{L} = \frac{3}{2} k^{1/2} \cdot \varepsilon^{-1} \cdot \dot{k} - k^{3/2} \cdot \varepsilon^{-2} \cdot \dot{\varepsilon} \tag{3.12}$$

Replacing the time derivatives of k and epsilon yield

$$
\begin{aligned}
\dot{L} &= \frac{3}{2} k^{1/2} \cdot \varepsilon^{-1} \cdot \left(C_\mu \cdot \frac{k^2}{\varepsilon} \cdot G - \varepsilon \right) - k^{3/2} \cdot \varepsilon^{-2} \cdot \left(C_{1\varepsilon} \cdot C_\mu \cdot k \cdot G - C_{2\varepsilon} \frac{\varepsilon^2}{k} \right) \\
\dot{L} &= \left(\frac{3}{2} - C_{1\varepsilon} \right) \cdot C_\mu \cdot \frac{k^{5/2}}{\varepsilon^2} \cdot G + \left(C_{2\varepsilon} - \frac{3}{2} \right) \cdot k^{1/2}
\end{aligned}
\tag{3.13}
$$

Division by the length scale itself gives

$$
\frac{\dot{L}}{L} = \left(\frac{3}{2} - C_{1\varepsilon} \right) \cdot C_\mu \cdot \frac{k}{\varepsilon} \cdot G + \left(C_{2\varepsilon} - \frac{3}{2} \right) \cdot \frac{\varepsilon}{k}
\tag{3.14}
$$

Therefore, for the nondimensionalized change rate of the turbulent length scale, we also find a limit as time goes to infinity:

$$
\begin{aligned}
\frac{\dot{L}}{L} &= \left(\frac{3}{2} - C_{1\varepsilon} \right) \cdot \sqrt{\frac{(1 - C_{2\varepsilon})}{(1 - C_{1\varepsilon})}} \cdot \sqrt{C_\mu \cdot G} + \left(C_{2\varepsilon} - \frac{3}{2} \right) \cdot \sqrt{\frac{(1 - C_{1\varepsilon})}{(1 - C_{2\varepsilon})}} \cdot \sqrt{C_\mu \cdot G} \\
\frac{\dot{L}}{L} &= \left(\left(\frac{3}{2} - C_{1\varepsilon} \right) \cdot \sqrt{\frac{(1 - C_{2\varepsilon})}{(1 - C_{1\varepsilon})}} + \left(C_{2\varepsilon} - \frac{3}{2} \right) \cdot \sqrt{\frac{(1 - C_{1\varepsilon})}{(1 - C_{2\varepsilon})}} \right) \cdot \sqrt{C_\mu \cdot G}
\end{aligned}
\tag{3.15}
$$

Equation (3.14) reveals a paradoxon as the term $\frac{\dot{L}}{L}$ is strictly positive for all G ($C_{1\varepsilon} = 1.44,\ C_{2\varepsilon} = 1.92$). **This means that the turbulent length scale rises even under high shear deformation of the fluid, which is not consistent in itself**. We would expect a shrinkage of this quantity under high shear rates.

Equation (3.15) an adaption of the value of $C_{1\varepsilon}$. As it states the time change rate of the length scale as time goes to infinity, we expect no change of the length scale anymore at that time, hence

$$
\left(\left(\frac{3}{2} - C_{1\varepsilon} \right) \cdot \sqrt{\frac{(1 - C_{2\varepsilon})}{(1 - C_{1\varepsilon})}} + \left(C_{2\varepsilon} - \frac{3}{2} \right) \cdot \sqrt{\frac{(1 - C_{1\varepsilon})}{(1 - C_{2\varepsilon})}} \right) \cdot \sqrt{C_\mu \cdot G} = 0
\tag{3.16}
$$

and hence

$$
C_{2\varepsilon} = C_{1\varepsilon}
\tag{3.17}
$$

3.5 Strategies of initializing (PUSHING) k and epsilon at high shear rates

Due to the paradoxon described in the section above, and also due to the fact that small starting values of k and epsilon might lead to extremely long times until reaching significant k and epsilon, we might want to initialize the values of k and epsilon at high shear rates.

3.5.1 Strategy 1: infinity ratio of of k and epsilon

Assume the infinity ratio of k and epsilon and assume an initial turbulent viscosity equal to some initial viscosity. I.e.

$$\frac{k_{init}}{\varepsilon_{init}} = \sqrt{\frac{(1-C_{2\varepsilon})}{(1-C_{1\varepsilon})}} \cdot \sqrt{\frac{1}{C_\mu \cdot G}} \tag{3.18}$$

Or in other words

$$\varepsilon_{init} = \sqrt{\frac{(1-C_{1\varepsilon})}{(1-C_{2\varepsilon})}} \cdot \sqrt{C_\mu \cdot G} \cdot k_{init} \quad = \quad 0.6916 \cdot \sqrt{C_\mu \cdot G} \cdot k_{init} \tag{3.19}$$

3.5.2 Strategy 2: zero change rate of epsilon

Let us assume $\dot{\varepsilon} = 0$ at the time of pushing. Thus, let us enforce

$$\begin{aligned} \dot{\varepsilon} &= 0 = C_{1\varepsilon} \cdot C_\mu \cdot k_{init} \cdot G - C_{2\varepsilon} \frac{\varepsilon_{init}^2}{k_{init}} \\ 0 &= C_{1\varepsilon} \cdot C_\mu \cdot k_{init}^2 \cdot G - C_{2\varepsilon} \cdot \varepsilon_{init}^2 \\ \varepsilon_{init} &= \sqrt{\frac{C_{1\varepsilon}}{C_{2\varepsilon}}} \cdot \sqrt{C_\mu \cdot G} \cdot k_{init} \quad = \quad 0.8660 \cdot \sqrt{C_\mu \cdot G} \cdot k_{init} \end{aligned} \tag{3.20}$$

We have

$$\begin{aligned} \frac{\varepsilon_{init}}{k_{init}} &= \sqrt{\frac{C_{1\varepsilon}}{C_{2\varepsilon}}} \cdot \sqrt{C_\mu \cdot G} \\ \frac{k_{init}}{\varepsilon_{init}} &= \sqrt{\frac{C_{2\varepsilon}}{C_{1\varepsilon}}} \cdot \frac{1}{\sqrt{C_\mu \cdot G}} \end{aligned} \tag{3.21}$$

3.5.3 Initialize k by initial-viscosity-assumption

Let us suppose we would like to provide an initial viscosity ν_{init} .

3.5.3.1 Follow strategy 1

In case of strategy 1, we have

$$\nu_{init} = C_\mu \cdot \frac{k_{init}}{\varepsilon_{init}} \cdot k_{init} = C_\mu \sqrt{\frac{(1-C_{2\varepsilon})}{(1-C_{1\varepsilon})}} \cdot \sqrt{\frac{C_\mu}{G}} \cdot k_{init}$$

$$k_{init} = \frac{\nu_{init}}{C_\mu \cdot \sqrt{\frac{(1-C_{2\varepsilon})}{(1-C_{1\varepsilon})}} \cdot \sqrt{\frac{1}{C_\mu \cdot G}}} = \frac{\sqrt{(1-C_{1\varepsilon})}}{\sqrt{(1-C_{2\varepsilon})}} \cdot \frac{\sqrt{G}}{\sqrt{C_\mu}} \cdot \nu_{init} = 0.6916 \cdot \frac{\sqrt{G}}{\sqrt{C_\mu}} \cdot \nu_{init}$$

$$\varepsilon_{init} = \frac{(1-C_{1\varepsilon})}{(1-C_{2\varepsilon})} G \cdot \nu_{init} = 0.4783 \cdot G \cdot \nu_{init}$$

$$L_{init} = \sqrt{C_\mu} \cdot \sqrt[4]{\frac{(1-C_{2\varepsilon})}{(1-C_{1\varepsilon})}} \cdot \sqrt[4]{\frac{1}{C_\mu G}} \cdot \nu_{init}^{1/2} = 0.3607 \cdot \frac{1}{\sqrt[4]{C_\mu G}} \cdot \nu_{init}^{1/2} \qquad (3.22)$$

The time change rate of k and epsilon are in that case

$$\frac{d}{dt}\ln(k) = \left(\sqrt{\frac{(1-C_{2\varepsilon})}{(1-C_{1\varepsilon})}} - \sqrt{\frac{(1-C_{1\varepsilon})}{(1-C_{2\varepsilon})}} \right) \cdot \sqrt{C_\mu \cdot G} = \left(\frac{(C_{2\varepsilon}-1)-(C_{1\varepsilon}-1)}{\sqrt{(1-C_{1\varepsilon})(1-C_{2\varepsilon})}} \right) \cdot \sqrt{C_\mu \cdot G}$$

$$\frac{d}{dt}\ln(\varepsilon) = \left(C_{1\varepsilon}\sqrt{\frac{(1-C_{2\varepsilon})}{(1-C_{1\varepsilon})}} - C_{2\varepsilon}\sqrt{\frac{(1-C_{1\varepsilon})}{(1-C_{2\varepsilon})}} \right) \cdot \sqrt{C_\mu \cdot G} = \left(\frac{C_{1\varepsilon}(C_{2\varepsilon}-1)-C_{2\varepsilon}(C_{1\varepsilon}-1)}{\sqrt{(1-C_{1\varepsilon})(1-C_{2\varepsilon})}} \right) \cdot \sqrt{C_\mu \cdot G}$$

(3.23)

Or more easy

$$\frac{d}{dt}\ln(\mathrm{k}) = \left(\frac{\mathrm{C}_{2\varepsilon} - \mathrm{C}_{1\varepsilon}}{\sqrt{\left(1-\mathrm{C}_{1\varepsilon}\right)\left(1-\mathrm{C}_{2\varepsilon}\right)}} \right) \cdot \sqrt{C_\mu \cdot \mathrm{G}} \quad = \quad 0.7544 \cdot \sqrt{C_\mu \cdot G}$$

$$\frac{d}{dt}\ln(\varepsilon) = \left(\frac{\mathrm{C}_{2\varepsilon} - \mathrm{C}_{1\varepsilon}}{\sqrt{\left(1-\mathrm{C}_{1\varepsilon}\right)\left(1-\mathrm{C}_{2\varepsilon}\right)}} \right) \cdot \sqrt{C_\mu \cdot \mathrm{G}} \quad = \quad 0.7544 \cdot \sqrt{C_\mu \cdot G} \tag{3.24}$$

$$\frac{d}{dt}\ln\left(\nu_t\right) = \left(\frac{\mathrm{C}_{2\varepsilon} - \mathrm{C}_{1\varepsilon}}{\sqrt{\left(1-\mathrm{C}_{1\varepsilon}\right)\left(1-\mathrm{C}_{2\varepsilon}\right)}} \right) \cdot \sqrt{C_\mu \cdot \mathrm{G}} \quad = \quad 0.7544 \cdot \sqrt{C_\mu \cdot G}$$

Let us assume we choose the initial viscosity to be some multiple of the Smagorinskij viscosity. That is

$$\nu_{init} = \alpha\left(c_s h\right)^2 \sqrt{G} \tag{3.25}$$

Then we have

$$\mathrm{k}_{init} = \alpha \cdot \frac{\sqrt{\left(1-\mathrm{C}_{1\varepsilon}\right)}}{\sqrt{\left(1-\mathrm{C}_{2\varepsilon}\right)}} \cdot \frac{1}{\sqrt{C_\mu}} \cdot \left(c_s h\right)^2 G$$

$$\varepsilon_{init} = \alpha \cdot \frac{\left(1-\mathrm{C}_{1\varepsilon}\right)}{\left(1-\mathrm{C}_{2\varepsilon}\right)} \cdot \left(c_s h\right)^2 \cdot \mathrm{G}^{3/2} \tag{3.26}$$

$$L_{init} = \sqrt{\alpha} \cdot \sqrt[4]{C_\mu \frac{\left(1-\mathrm{C}_{2\varepsilon}\right)}{\left(1-\mathrm{C}_{1\varepsilon}\right)}} \cdot \left(c_s h\right) = \sqrt{\alpha} \cdot 0.6586 \cdot \left(c_s h\right)$$

3.5.3.2 Follow strategy 2

$$\nu_{init} = C_\mu \cdot \frac{\mathrm{k}_{init}}{\varepsilon_{init}} \cdot \mathrm{k}_{init} = \sqrt{\frac{C_{2\varepsilon}}{C_{1\varepsilon}} \frac{\sqrt{C_\mu}}{\sqrt{G}}} \cdot \mathrm{k}_{init}$$

$$\mathrm{k}_{init} = \frac{\nu_{init}}{\sqrt{\frac{C_{2\varepsilon}}{C_{1\varepsilon}} \frac{\sqrt{C_\mu}}{\sqrt{G}}}} = \sqrt{\frac{C_{1\varepsilon}}{C_{2\varepsilon}} \cdot \frac{\sqrt{\mathrm{G}}}{\sqrt{C_\mu}}} \cdot \nu_{init} = 0.8660 \cdot \frac{\sqrt{\mathrm{G}}}{\sqrt{C_\mu}} \cdot \nu_{init} \tag{3.27}$$

$$\varepsilon_{init} = \sqrt{\frac{C_{1\varepsilon}}{C_{2\varepsilon}}} \cdot \sqrt{C_\mu \cdot G} \cdot \sqrt{\frac{C_{1\varepsilon}}{C_{2\varepsilon}} \cdot \frac{\sqrt{\mathrm{G}}}{\sqrt{C_\mu}}} \cdot \nu_{init} = \frac{C_{1\varepsilon}}{C_{2\varepsilon}} \cdot G \cdot \nu_{init} = 0.75 \cdot G \cdot \nu_{init}$$

$$L_{init} = C_\mu \frac{k_{init}}{\varepsilon_{init}} k_{init}^{1/2} = C_\mu \cdot \frac{1}{\sqrt{C_\mu}} \cdot \nu_{init}^{1/2} = \sqrt{C_\mu} \sqrt[4]{\frac{C_{2\varepsilon}}{C_{1\varepsilon}}} \cdot \frac{1}{\sqrt[4]{C_\mu \cdot G}} \cdot \nu_{init}^{1/2} = 0.3224 \cdot \frac{1}{\sqrt[4]{C_\mu \cdot G}} \cdot \nu_{init}^{1/2}$$

And the change rates of the quantities are

$$\frac{d}{dt}\ln(\mathrm{k}) = \mathrm{C}_\mu \cdot \sqrt{\frac{C_{2\varepsilon}}{C_{1\varepsilon}}} \cdot \frac{1}{\sqrt{C_\mu \cdot G}} \cdot \mathrm{G} - \sqrt{\frac{C_{1\varepsilon}}{C_{2\varepsilon}}} \cdot \sqrt{C_\mu \cdot G} = \left(\sqrt{\frac{C_{2\varepsilon}}{C_{1\varepsilon}}} - \sqrt{\frac{C_{1\varepsilon}}{C_{2\varepsilon}}} \right) \cdot \sqrt{C_\mu \cdot G} = 0.2887 \cdot \sqrt{C_\mu \cdot G}$$

$$\frac{d}{dt}\ln(\varepsilon) = \mathrm{C}_{1\varepsilon} \cdot \mathrm{C}_\mu \cdot \sqrt{\frac{C_{2\varepsilon}}{C_{1\varepsilon}}} \cdot \frac{1}{\sqrt{C_\mu \cdot G}} \cdot \mathrm{G} - \mathrm{C}_{2\varepsilon} \sqrt{\frac{C_{1\varepsilon}}{C_{2\varepsilon}}} \cdot \sqrt{C_\mu \cdot G} = \left(\mathrm{C}_{1\varepsilon} \cdot \sqrt{\frac{C_{2\varepsilon}}{C_{1\varepsilon}}} - \mathrm{C}_{2\varepsilon} \sqrt{\frac{C_{1\varepsilon}}{C_{2\varepsilon}}} \right) \cdot \sqrt{C_\mu \cdot G} = 0$$

$$\frac{d}{dt}\ln(\nu) = \frac{2\left(C_{2\varepsilon} - C_{1\varepsilon}\right)}{\sqrt{C_{1\varepsilon} C_{2\varepsilon}}} \cdot \sqrt{C_\mu \cdot G} = 0.5774 \cdot \sqrt{C_\mu \cdot G}$$

(3.28)

3.5.4 Provide initial value of k

Here, we assume a certain initial value for k by

$$\mathrm{k}_{init} = \alpha \cdot \mathrm{h}^2 \cdot \mathrm{G} \qquad (3.29)$$

3.5.4.1 Follow strategy 1

$$k_{init} = \alpha \cdot h^2 \cdot G$$

$$\varepsilon_{init} = \sqrt{\frac{\left(1 - C_{1\varepsilon}\right)}{\left(1 - C_{2\varepsilon}\right)}} \cdot \sqrt{C_\mu \cdot G} \cdot k_{init} = \alpha \cdot \sqrt{C_\mu} \cdot \sqrt{\frac{\left(1 - C_{1\varepsilon}\right)}{\left(1 - C_{2\varepsilon}\right)}} \cdot h^2 \cdot G^{3/2} = \; = \alpha \cdot 0.2074 \cdot h^2 \cdot G^{3/2}$$

$$\nu_{init} = C_\mu \cdot \frac{k_{init}}{\varepsilon_{init}} \cdot k_{init} = \alpha \cdot \sqrt{C_\mu \cdot \frac{\left(1 - C_{2\varepsilon}\right)}{\left(1 - C_{1\varepsilon}\right)}} \cdot h^2 \cdot \sqrt{G} = \alpha \cdot 0.4338 \cdot h^2 \cdot \sqrt{G}$$

$$L_{init} = C_\mu \cdot \frac{k_{init}}{\varepsilon_{init}} \cdot k_{init}^{1/2} = C_\mu \cdot \sqrt{\frac{\left(1 - C_{2\varepsilon}\right)}{\left(1 - C_{1\varepsilon}\right)}} \cdot \sqrt{\frac{1}{C_\mu \cdot G}} \cdot \sqrt{\alpha \cdot h^2 \cdot G} = \sqrt{\alpha} \cdot \sqrt{C_\mu} \sqrt{\frac{\left(1 - C_{2\varepsilon}\right)}{\left(1 - C_{1\varepsilon}\right)}} \cdot h = \sqrt{\alpha} \cdot 0.4338 \cdot h$$

(3.30)

With values for $\alpha = 0.001 ... 1$ or so.

If we choose $c_S^2 = \alpha \cdot 0.4338 \;\Rightarrow\; \alpha = \frac{c_S^2}{0.4338} = \frac{0.1^2}{0.4338} = 0.02305$ then we find

$$\nu_t = \left(\mathrm{c}_S \mathrm{h}\right)^2 \cdot \sqrt{\mathrm{G}} \qquad (3.31)$$

which is exactly the turbulent viscosity proposed be the Smagorinsky-Turbulence model. In this way, for turbulent production, Smagorinsky and k-epsilon are similar. They are different in the decay of the turbulences.

If we choose $\sqrt{C_\mu}\sqrt{\frac{(1-C_{2\varepsilon})}{(1-C_{1\varepsilon})}}\cdot\sqrt{\alpha}<\frac{1}{2} \quad\Rightarrow\quad \alpha<\frac{1}{4}\frac{1}{C_\mu}\frac{(1-C_{1\varepsilon})}{(1-C_{2\varepsilon})}=1.3285$, we can be sure that the produced turbulent length scale is small enough (not bigger than the approximate point distance).

In general, by choosing alpha we have the freedom to either influence the turbulent viscosity or the turbulent length scale to a desired value.

3.5.4.2 Follow Strategy 2

Let us assume $\dot{\varepsilon}=0$ at the time of pushing. Like above, we have

$$k_{init}=\alpha\cdot h^2\cdot G \tag{3.32}$$

and that leads to

$$\begin{aligned}
\varepsilon_{init} &= \sqrt{\frac{C_{1\varepsilon}}{C_{2\varepsilon}}}\cdot\sqrt{C_\mu\cdot G}\cdot k_{init} = \alpha\cdot\sqrt{\frac{C_{1\varepsilon}}{C_{2\varepsilon}}}\cdot\sqrt{C_\mu}\cdot h^2\cdot G^{3/2} = \alpha\cdot 0.2598\cdot h^2\cdot G^{3/2}\\
\nu_{init} &= C_\mu\frac{k_{init}}{\varepsilon_{init}}k_{init} = \alpha\cdot\sqrt{C_\mu}\sqrt{\frac{C_{2\varepsilon}}{C_{1\varepsilon}}}\cdot h^2\cdot\sqrt{G} = \alpha\cdot 0.3464\cdot h^2\cdot\sqrt{G}\\
L_{init} &= C_\mu\frac{k_{init}}{\varepsilon_{init}}k_{init}^{1/2} = \sqrt{\alpha}\cdot\sqrt{C_\mu}\sqrt{\frac{C_{2\varepsilon}}{C_{1\varepsilon}}}h = \sqrt{\alpha}\cdot 0.3464\cdot h
\end{aligned} \tag{3.33}$$

And we have

$$\begin{aligned}
\frac{d}{dt}\ln(k) &= \left(\frac{C_{2\varepsilon}-C_{1\varepsilon}}{\sqrt{C_{1\varepsilon}C_{2\varepsilon}}}\right)\cdot\sqrt{C_\mu\cdot G} = 0.2887\cdot\sqrt{C_\mu\cdot G}\\
\frac{d}{dt}\ln(\varepsilon) &= 0\\
\frac{d}{dt}\ln(\nu) &= 2\cdot\frac{(C_{2\varepsilon}-C_{1\varepsilon})}{\sqrt{C_{1\varepsilon}C_{2\varepsilon}}}\cdot\sqrt{C_\mu\cdot G} = 0.5774\cdot\sqrt{C_\mu\cdot G}
\end{aligned} \tag{3.34}$$

3.6 Strategies of Zerofying k and epsilon

As k and epsilon must not be set to zero, the question is, what is a good/acceptable small value for k end epsilon (such as non-turbulent inflow).

First we want, that the initialized k does not decay too rapidly, that is

$$-\frac{dk}{dt} = \varepsilon_{init} < D \cdot \frac{k_{init}}{\Delta t} \quad \Rightarrow \quad \frac{\varepsilon_{init}}{k_{init}} < \frac{D}{\Delta t} \tag{3.35}$$

Secondly we want that the resulting turbulent viscosity is smaller that the physical viscosity, that is

$$c_\mu \frac{k_{init}^2}{\varepsilon_{init}} < A \cdot \nu_{phys} \quad \Rightarrow \quad \frac{c_\mu}{A \cdot \nu_{phys}} k_{init} < \frac{\varepsilon_{init}}{k_{init}} \tag{3.36}$$

Hence it follows

$$\begin{aligned} &\frac{c_\mu}{A \cdot \nu_{phys}} k_{init} < \frac{\varepsilon_{init}}{k_{init}} < \frac{D}{\Delta t} \\ &k_{init} < \frac{D}{\Delta t} \frac{A \cdot \nu_{phys}}{c_\mu} \\ &\varepsilon_{init} = \frac{c_\mu}{A \cdot \nu_{phys}} k_{init}^2 \end{aligned} \tag{3.37}$$

Choose D=0.1…0.01 as well as A=0.1…0.01

4 K-omega turbulence model

4.1 Consistency of k-omega to k-epsilon model

It seems, that the consistency to the k-epsilon model is given through

$$\omega = \frac{1}{c_\mu}\frac{\varepsilon}{k} \quad \Leftrightarrow \quad \frac{k}{\varepsilon} = \frac{1}{c_\mu}\frac{1}{\omega} \quad \Leftrightarrow \quad \frac{\varepsilon}{k} = c_\mu \omega \tag{4.1}$$

provided that $\beta^* = c_\mu$, which is the case, as $\beta^* = c_\mu = 0.09$ in independent papers.

4.2 Standard model by Wilcox

The k-omega model by Wilcox is

$$\begin{aligned} \frac{d(\rho k)}{dt} &= \nabla^T\left(\left(\eta + \frac{\eta_{turb}}{\sigma_k}\right)\nabla k\right) - \beta^* \rho\omega k + P \\ \frac{d(\rho\omega)}{dt} &= \nabla^T\left(\left(\eta + \frac{\eta_{turb}}{\sigma_\omega}\right)\nabla \omega\right) + \frac{\rho\sigma_d}{\omega}(\nabla k)^T \cdot (\nabla\omega) - \beta\rho\omega^2 + \gamma\frac{\omega}{k}\cdot P \end{aligned} \tag{4.2}$$

The basic constants are stated for example in
https://turbmodels.larc.nasa.gov/wilcox.html by

$$\sigma_k = \frac{1}{0.6} = 1.66$$
$$\sigma_\omega = \frac{1}{0.5} = 2.0$$
$$\beta^* = 0.09$$
$$\beta = \beta_0 \cdot f_\beta$$
$$\beta_0 = 0.0708$$
$$\gamma = \frac{13}{25} = 0.515$$
$$C_{\lim} = \frac{7}{8}$$

The value of f_β is computed by

$$f_\beta = \frac{1+85\chi_\omega}{1+100\chi_\omega}$$

$$\chi_\omega = \left|\frac{\Omega_{ij}\Omega_{jk}\hat{S}_{ki}}{\beta^*\omega^3}\right| = \left|\Omega_{ij}\Omega_{jk}\hat{S}_{ki}\frac{\left(c_\mu k\right)^3}{\left(\beta^*\varepsilon\right)^3}\right| \tag{4.3}$$

$$\mathbf{\Omega} = \frac{1}{2}\left(\nabla\mathbf{v}^T - \left(\nabla\mathbf{v}^T\right)^T\right)$$

$$\hat{\mathbf{S}} = \frac{1}{2}\left(\nabla\mathbf{v}^T + \left(\nabla\mathbf{v}^T\right)^T\right) - \frac{1}{2}\left(\nabla^T\mathbf{v}\right)\mathbf{I}$$

The value of σ_d is given by

$$\sigma_d = \begin{cases} 0 & \text{if} \quad \left(\nabla k\right)^T \cdot \left(\nabla\omega\right) \le 0 \\ \frac{1}{8} & \text{if} \quad \left(\nabla k\right)^T \cdot \left(\nabla\omega\right) > 0 \end{cases} \tag{4.4}$$

4.2.1 Computation of the turbulent viscosity

$$\nu_t = \frac{k}{\hat{\omega}} \qquad \hat{\omega} = \max\left[\omega, C_{\lim}\sqrt{\frac{2\bar{S}_{ij}\bar{S}_{ij}}{\beta^*}}\right] \qquad \bar{\mathbf{S}} = \frac{1}{2}\left(\nabla\mathbf{v}^T + \left(\nabla\mathbf{v}^T\right)^T\right) - \frac{1}{3}\left(\nabla^T\mathbf{v}\right)\mathbf{I} \tag{4.5}$$

4.2.2 Consistency of the turbulent viscosity

With equation (4.1) and the definition of the turbulent viscosity in the k-epsilon model, we can map the viscosity over from k-omega to the k-epsilon model. We obtain an adapted value for $c_\mu^{k\omega}$

$$c_\mu^{k\omega}\frac{k^2}{\varepsilon} = \frac{k}{\hat{\omega}} \quad \Rightarrow \quad c_\mu^{k\omega} = \frac{\varepsilon}{k}\frac{1}{\hat{\omega}} \quad \Rightarrow \quad c_\mu^{k\omega} = \frac{\varepsilon}{k}\frac{1}{\max\left[\omega, C_{\lim}\sqrt{\frac{2\bar{S}_{ij}\bar{S}_{ij}}{\beta^*}}\right]}$$

$$c_\mu^{k\omega} = \frac{\varepsilon}{k}\frac{1}{\max\left[\frac{1}{c_\mu}\frac{\varepsilon}{k}, C_{\lim}\sqrt{\frac{2\bar{S}_{ij}\bar{S}_{ij}}{\beta^*}}\right]} \quad \Rightarrow \quad c_\mu^{k\omega} = \frac{\varepsilon}{k}\frac{1}{\frac{1}{c_\mu}\max\left[\frac{\varepsilon}{k}, c_\mu C_{\lim}\sqrt{\frac{2\bar{S}_{ij}\bar{S}_{ij}}{\beta^*}}\right]} \tag{4.6}$$

$$c_\mu^{k\omega} = c_\mu\frac{\frac{\varepsilon}{k}}{\max\left[\frac{\varepsilon}{k}, c_\mu C_{\lim}\sqrt{\frac{2\bar{S}_{ij}\bar{S}_{ij}}{\beta^*}}\right]} = c_\mu\frac{1}{\max\left[1, \frac{c_\mu}{\sqrt{\beta^*}}C_{\lim}\frac{k}{\varepsilon}\sqrt{2\bar{S}_{ij}\bar{S}_{ij}}\right]}$$

4.2.3 Consistency of the values C1 and C2

Let us consider again the considerations in the k-epsilon model concerning the term $\frac{k}{\varepsilon}$. Equation (1.7) states the time evolution of this term, which, by virtue of (4.1), can be rewritten as

$$\frac{\mathrm{d}}{\mathrm{dt}}\left(c_\mu \omega\right) = \left(1-\mathrm{C}_{2\varepsilon}\right)\cdot\left(c_\mu \omega\right)^2 + \mathrm{c}_\mu\left(\mathrm{C}_{1\varepsilon}-1\right)\|\mathbf{D}\|_M^2 \qquad (4.7)$$

Transformation of the equation gives

$$\frac{\mathrm{d}\omega}{\mathrm{dt}} = c_\mu\left(1-\mathrm{C}_{2\varepsilon}\right)\cdot\omega^2 + \left(\mathrm{C}_{1\varepsilon}-1\right)\|\mathbf{D}\|_M^2 \qquad (4.8)$$

The appropriate equation from the model (4.1) is

$$\frac{d\omega}{dt} = \nabla^T\left(\left(\nu+\frac{\nu_{turb}}{\sigma_\omega}\right)\nabla\omega\right) - \beta\omega^2 + \gamma\frac{\omega}{k}\cdot\frac{1}{\rho}P \qquad (4.9)$$

And under assumption of constant distribution of omega, and with $P=\rho\nu_{turb}\|D\|^2$ we get

$$\begin{aligned}
\frac{d\omega}{dt} &= -\beta\omega^2 + \gamma\frac{\omega}{k}\cdot\nu_{turb}\|D\|^2 \\
\frac{d\omega}{dt} &= -\beta\omega^2 + \gamma\frac{\varepsilon}{c_\mu k^2}\cdot c_\mu^{k\omega}\frac{k^2}{\varepsilon}\|D\|^2 \\
\frac{d\omega}{dt} &= -\beta\omega^2 + \gamma\frac{c_\mu^{k\omega}}{c_\mu}\|D\|^2
\end{aligned} \qquad (4.10)$$

Comparing (4.10) and (4.8) yields the consistency criteria

$$\begin{aligned}
c_\mu\left(\mathrm{C}_{2\varepsilon}-1\right) &= \beta \qquad && 0.0828 \neq 0.0707 \\
\left(\mathrm{C}_{1\varepsilon}-1\right) &= \gamma\frac{c_\mu^{k\omega}}{c_\mu} \qquad && 0.44 \neq 0.515
\end{aligned} \qquad (4.11)$$

Or in other words

$$\mathrm{C}_{2\varepsilon}^{k\omega} = \frac{\beta}{c_\mu} + 1 \qquad 1.92 \neq 1.7866 \quad \text{if no gradient of k and omega}$$
$$\mathrm{C}_{1\varepsilon}^{k\omega} = \gamma \frac{c_\mu^{k\omega}}{c_\mu} + 1 \qquad 1.44 \neq 1.515 \quad \text{if no gradient of k and omega} \tag{4.12}$$

4.2.4 Treatment of additional sources in the k-omega-model

Let us find out how to treat additional sources in the k-omega-model under projection into k-epsilon model.

We have

$$\varepsilon = c_\mu k\omega$$
$$\frac{d\varepsilon}{dt} = c_\mu \left(\frac{dk}{dt}\omega + k\frac{d\omega}{dt} \right) \tag{4.13}$$
$$\frac{d\varepsilon}{dt} + \left[\frac{d\varepsilon}{dt} \right]_{add} = c_\mu \left(\frac{dk}{dt}\omega + \left[\frac{dk}{dt} \right]_{add} \omega + k\frac{d\omega}{dt} + k\left[\frac{d\omega}{dt} \right]_{add} \right)$$

Let us assume that k does not change significantly due to the additional source. Then

$$\frac{d\varepsilon}{dt} + \left[\frac{d\varepsilon}{dt} \right]_{add} = c_\mu \left(\frac{dk}{dt}\omega + k\frac{d\omega}{dt} + k\left[\frac{d\omega}{dt} \right]_{add} \right) \tag{4.14}$$

And we can decouple

$$\frac{d\varepsilon}{dt} = c_\mu \left(\frac{dk}{dt}\omega + k\frac{d\omega}{dt} \right) \qquad \text{regular changes}$$
$$\left[\frac{d\varepsilon}{dt} \right]_{add} = c_\mu \cdot k\left[\frac{d\omega}{dt} \right]_{add} \qquad \text{additional changes} \tag{4.15}$$

4.2.4.1 Special case gradient k scalar gradient omega

Let us look at the term

$$\left[\frac{d\omega}{dt} \right]_{add} = \frac{\sigma_d}{\omega} (\nabla k)^T \cdot (\nabla \omega) \tag{4.16}$$

That turns into

$$\left[\frac{d\varepsilon}{dt}\right]_{add} = c_\mu \cdot k_{ref} \left[\frac{d\omega}{dt}\right]_{add} \tag{4.17}$$

Which is

$$\begin{aligned}
\left[\frac{d\varepsilon}{dt}\right]_{add} &= c_\mu \cdot k_{ref} \frac{\sigma_d}{\omega} (\nabla k)^T \cdot (\nabla \omega) \\
\left[\frac{d\varepsilon}{dt}\right]_{add} &= c_\mu \cdot k_{ref} \frac{\sigma_d}{\omega} (\nabla k)^T \cdot \left(\nabla \left(\frac{1}{c_\mu}\frac{\varepsilon}{k}\right)\right) = k_{ref} \frac{\sigma_d}{\omega_{ref}} (\nabla k)^T \cdot \left(\nabla \left(\frac{\varepsilon}{k}\right)\right) \\
\left[\frac{d\varepsilon}{dt}\right]_{add} &= k \frac{\sigma_d}{\omega} (\nabla k)^T \cdot \left(\frac{1}{k}\nabla\varepsilon - \left(\frac{\varepsilon}{k^2}\right)\nabla k\right) \\
\left[\frac{d\varepsilon}{dt}\right]_{add} &= \frac{\sigma_d}{\omega} (\nabla k)^T \cdot \nabla\varepsilon - \frac{\varepsilon}{k}\frac{\sigma_d}{\omega} (\nabla k)^T \cdot \nabla k
\end{aligned} \tag{4.18}$$

Further simplification yields

$$\begin{aligned}
\left[\frac{d\varepsilon}{dt}\right]_{add} &= c_\mu \frac{k}{\varepsilon} \sigma_d \left((\nabla k)^T \cdot \nabla\varepsilon\right) - c_\mu \sigma_d \left((\nabla k)^T \cdot \nabla k\right) \\
\left[\frac{d\varepsilon}{dt}\right]_{add} &= c_\mu \sigma_d k^2 \left((\nabla \log k)^T \cdot \nabla \log\varepsilon\right) - c_\mu \sigma_d k^2 \left((\nabla \log k)^T \cdot \nabla \log k\right)
\end{aligned} \tag{4.19}$$

4.3 The Menter SST model

4.3.1 Consistency of the turbulent viscosity

$$\begin{aligned}
\nu_t &= \frac{a_1 k}{\max\left(a_1 \omega, F_2 S\right)} \\
\nu_t &= \frac{a_1 k}{\max\left(a_1 \frac{\varepsilon}{c_\mu k}, F_2 S\right)} = \frac{a_1 k}{\max\left(\frac{a_1}{c_\mu}\frac{\varepsilon}{k}, F_2 S\right)} = \frac{a_1 k}{\frac{a_1}{c_\mu}\frac{\varepsilon}{k}\max\left(1, \frac{c_\mu}{a_1}\frac{k}{\varepsilon} F_2 S\right)} \\
\nu_t &= \frac{k^2}{\varepsilon}\frac{c_\mu}{\max\left(1, \frac{c_\mu}{a_1}\frac{k}{\varepsilon} F_2 S\right)}
\end{aligned} \tag{4.20}$$

And hence the derived c_mue value is

$$c_\mu^{k\omega} = \frac{c_\mu}{\max\left(1, \frac{c_\mu}{a_1}\frac{k}{\varepsilon}F_2\, S\right)} \tag{4.21}$$

4.3.2 Consistency of G

The production term is restricted in Menter SST model in the sense

$$\begin{aligned} \tilde{P} &= \min\left(\nu_t\, G, 10\beta^* k\omega\right) = \min\left(\nu_t\, G, 10\beta^* k \frac{\varepsilon}{c_\mu k}\right) \\ &= \min\left(\nu_t\, G, 10\frac{\beta^*}{c_\mu}\varepsilon\right) = \nu_t\, \min\left(G, 10\frac{\beta^*}{c_\mu}\frac{\varepsilon}{\nu_t}\right) \end{aligned} \tag{4.22}$$

Thus, for our implementation, we can restrict G in the sense

$$\tilde{G} = \min\left(G, 10\frac{\beta^*}{c_\mu}\frac{\varepsilon^2}{c_\mu^{k\omega}k^2}\right) = \min\left(G, 10\frac{\beta^*}{c_\mu c_\mu^{k\omega}}\frac{\varepsilon^2}{k^2}\right) \tag{4.23}$$

4.3.3 Consistency of production-reduction in different notations

The production reduction terms are stated in literature in different ways, either using α (CFD-online and Altair) or γ (NASA and others). Consistency between these two terms is given by

$$\begin{aligned} & Q = -\beta\omega^2 \;+\; \gamma\frac{\omega}{k}\cdot P = -\beta\omega^2 \;+\; \alpha\cdot G \\ & \gamma\frac{\omega}{k}\cdot\nu_t\, G = \alpha\cdot G \\ & \gamma\frac{\varepsilon}{c_\mu^{k\omega}k^2}\cdot c_\mu\frac{k^2}{\varepsilon} = \alpha \\ & \gamma\frac{c_\mu}{c_\mu^{k\omega}} = \alpha \end{aligned} \tag{4.24}$$

We have

$$\begin{aligned} \gamma &= F_1\,\gamma_1 + (1-F_1)\gamma_2 \\ \alpha &= F_1\,\alpha_1 + (1-F_1)\alpha_2 \end{aligned} \tag{4.25}$$

With the values

$$\gamma_1 = \frac{\beta_1}{\beta^*} - \frac{\sigma_1^{\omega}\kappa^2}{\sqrt{\beta^*}} = 0.5532 \qquad \gamma_2 = \frac{\beta_2}{\beta^*} - \frac{\sigma_2^{\omega}\kappa^2}{\sqrt{\beta^*}} = 0.4404 \tag{4.26}$$

And

$$\alpha_1 = \frac{5}{9} = 0.5555 \qquad \alpha_2 = 0.44 \tag{4.27}$$

4.3.4 Consistency for the computation of F_1

For the computation of F_1 , the computation of the term CD is essential:

$$CD = 2\frac{\sigma_\omega}{\omega}(\nabla k)^T \cdot (\nabla \omega) = 2\frac{\sigma_\omega}{\omega}(\nabla k)^T \cdot (\nabla \omega)$$

$$CD = 2\frac{\sigma_\omega}{\omega}(\nabla k)^T \cdot \left(\nabla\left(\frac{1}{c_\mu}\frac{\varepsilon}{k}\right)\right) = 2\frac{\sigma_\omega}{\omega}\frac{1}{c_\mu}(\nabla k)^T \cdot \left(\nabla\left(\frac{\varepsilon}{k}\right)\right)$$

$$CD = 2\frac{\sigma_\omega}{\omega}\frac{1}{c_\mu}(\nabla k)^T \cdot \left(\frac{1}{k}\nabla\varepsilon - \left(\frac{\varepsilon}{k^2}\right)\nabla k\right) = 2\frac{\sigma_\omega}{\varepsilon}\frac{1}{c_\mu}c_\mu k(\nabla k)^T \cdot \left(\frac{1}{k}\nabla\varepsilon - \left(\frac{\varepsilon}{k^2}\right)\nabla k\right)$$

$$CD = 2\frac{\sigma_\omega}{\varepsilon}(\nabla k)^T \cdot \nabla\varepsilon - 2\frac{\sigma_\omega}{k}(\nabla k)^T \cdot \nabla k$$

$$CD = 2\sigma_\omega\left((\nabla k)^T \cdot \nabla\varepsilon\frac{1}{\varepsilon} - (\nabla k)^T \cdot \nabla k\frac{1}{k}\right)$$